Barbălat Lemma and its application in analysis of system stability

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The Chinese version is attached after the English version

***Abstract*—A set of primary formulations of Barbălat Lemma and its simple alternatives are summarized. The relationships among those formulations and their applicable scopes are investigated. The applications of Barbălat Lemma in analyzing asymptotic convergence of the system, adaptive control design and L_p stability are discussed via three examples.**

***Index Terms*—Barbălat Lemma, nonlinear systems, Lyapunov theory, asymptotic convergence**

I. Introduction

As is well known, stability is the most important performance specification of various control systems. Therefore, developing methods for effectively analyzing and determining the stability of control systems has always been a hot topic in control theory research [1]–[8]. In the late 19th century, Russian mathematician A. M. Lyapunov proposed the famous Lyapunov theory. This theory, with the emergence of control theory, has become the main method for analyzing and investigating the stability of control systems, and has gradually matured in the latter half of last century [1]–[4].

Without requiring exact solution of the system, Lyapunov theory determines the stability of system through the qualitative analysis of a function similar to the energy function. So far, the Lyapunov theory has long been the most general and effective method for investigating and determining the stability of the systems, especially for nonlinear systems. However, classical Lyapunov stability theory exists limitations. For one example, it is difficult to construct Lyapunov functions whose derivatives are negative definite (or negative semidefinite). For another example, when the derivative of Lyapunov function is negative semidefinite, asymptotic stability cannot be obtained, which sometimes cannot meet certain practical requirements.

LaSalle invariance principle overcomes some limitations of classical Lyapunov theory, which enables the asymptotic convergence of nonlinear autonomous (time-invariant) systems to be obtained even when the derivatives of Lyapunov-like functions are negative semidefinite [2]. However, LaSalle invariance principle is not applicable to the analysis of asymptotic convergence of nonlinear nonautonomous (time-varying) systems.

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Barbălat Lemma makes up for the deficiency of LaSalle invariance principle [6]. This lemma is a purely mathematical result concerning the asymptotic behavior of functions and their derivatives. Even so, if we can apply the lemma appropriately, such as finding Lyapunov-like functions with negative semidefinite derivatives, then the asymptotic convergence can be obtained for nonlinear nonautonomous systems. Nowadays, Barbălat Lemma plays an increasingly important role in control theory, especially in adaptive control theory [7]. While Barbălat Lemma has its basic and purely mathematical expression, it has become more diverse and been enriched with the development of control theory, to better analyze and study the asymptotic convergence of various nonlinear and nonautonomous systems [6], [9], [10].

II. Barbălat Lemma

Although Barbălat Lemma in this section is presented for the single variable, it is also applicable to multiple variables.

A. The Primary Formulations of Barbălat Lemma

Barbălat Lemma has been widely applied in the stability analysis of control systems since its proposal. The most commonly used formulation of this lemma is as follows:

Lemma 1: [6] Suppose $x : [0,\infty) \to \mathbf{R}$ is continuously differentiable and $\lim_{t\to\infty} x(t)$ exists. If $\dot{x}(t)$ is uniformly continuous on $[0,\infty)$, then $\lim_{t\to\infty} \dot{x}(t) = 0$.

If $\ddot{x}(t)$ exists and is bounded, then the uniform continuity of $\dot{x}(t)$ in Lemma 1 can be replaced by the boundedness of $\ddot{x}(t)$ to obtain the following formulation of Barbălat Lemma.

Lemma 2: [6] Suppose $x : [0,\infty) \to \mathbf{R}$ is continuously differentiable and $\lim_{t\to\infty} x(t)$ exists. If $\ddot{x}(t)$ exists and is bounded on $[0,\infty)$, then $\lim_{t\to\infty} \dot{x}(t) = 0$.

The following corollary is obvious.

Corollary 1: If $x : [0,\infty) \to \mathbf{R}$ is uniformly continuous, and $\lim_{t\to\infty} \int_0^t x(\tau)\mathrm{d}\tau$ exists and is bounded, then $\lim_{t\to\infty} x(t) = 0$.

B. Several Alternatives of Barbălat Lemma

The primary formulation of Barbălat Lemma can determine the asymptotic convergence of systems to a certain extent. Nevertheless, the formulation has some limitations in practical applications because it is difficult to combine with Lyapunov theory. To this end, by extending and deforming the primary formulation of Barbălat Lemma, we obtain the following formulations of Barbălat lemma.

Lemma 3: If $x : [0, \infty) \to \mathbf{R}$ is uniformly continuous and there exists $p \in [1, \infty)$ such that $x \in L_p$*, then $\lim_{t\to\infty} x(t) = 0$.

Proof. When $p = 1$, from $\big||x_2| - |x_1|\big| \leq |x_2 - x_1|$ and the uniform continuity of x, it follows that $|x|$ is also uniformly continuous. Let $F(t) = \int_0^t |x(\tau)|\mathrm{d}\tau, t \geq 0$. Then, applying Lemma 1, we have $\dot{F}(t) = |x(t)|$, and in turn the convergence of $x(t)$.

When $p > 1$, suppose by contradiction that $\lim_{t\to\infty} x(t) = 0$ does not hold. Then there exists a constant $\varepsilon_0 > 0$ such that for any $T > 0$, there is $t_T > 0$ such that $|x(t_T)| \geq \varepsilon_0$. Based on this, we can get an infinite time sequence $\Omega = \{t_i, i = 1, 2, \cdots\}$ such that $|x(t_i)| \geq \varepsilon_0, \forall t_i \in \Omega$. Since $x(t)$ is uniformly continuous, for given ε_0, there exists $\eta(\varepsilon_0) > 0$ such that for any $t', t'' \in \{t', t'' : |t'-t''| \leq \eta, t', t'' \in [0, \infty)\}$, there is

$$|x(t') - x(t''))| \leq \frac{\varepsilon_0}{2}.$$

Then, for any $t \in B_\eta \triangleq \{t : |t - t_i| \leq \eta, t \in [0, \infty), t_i \in \Omega\}$, there is

$$\begin{aligned} |x(t)| &= |x(t_i) + x(t) - x(t_i)| \\ &\geq |x(t_i)| - |x(t) - x(t_i)| \geq \frac{\varepsilon_0}{2}, \end{aligned}$$

that is,

$$|x(t)|^p \geq \frac{\varepsilon_0^p}{2^p}, \quad \forall t \in B_\eta.$$

The continuity of $x(t)$ implies that $x(t)$ always keeps positive or negative on B_η. So for all $t_i \in \Omega$, there is

$$\begin{aligned} &\Big| \int_0^{t_i+\eta} |x(t)|^p \mathrm{d}t - \int_0^{t_i-\eta} |x(t)|^p \mathrm{d}t \Big| \\ &= \int_{t_i-\eta}^{t_i+\eta} |x(t)|^p \mathrm{d}t \geq \int_{t_i-\eta}^{t_i+\eta} \frac{\varepsilon_0^p}{2^p} \mathrm{d}t \geq 2\eta \frac{\varepsilon_0^p}{2^p}, \end{aligned}$$

which means

$$\lim_{t_i\to\infty, t_i\in\Omega} \int_{t_i-\eta}^{t_i+\eta} |x(t)|^p \mathrm{d}t \geq 2\eta \frac{\varepsilon_0^p}{2^p}. \tag{1}$$

However, noting $\int_0^\infty |x(t)|^p \mathrm{d}t = p_\infty < \infty$, we have

$$\begin{aligned} &\lim_{t_i\to\infty, t_i\in\Omega} \int_{t_i-\eta}^{t_i+\eta} |x(t)|^p \mathrm{d}t \\ &= \lim_{t_i\to\infty, t_i\in\Omega} \int_0^{t_i+\eta} |x(t)|^p \mathrm{d}t - \lim_{t_i\to\infty, t_i\in\Omega} \int_0^{t_i-\eta} |x(t)|^p \mathrm{d}t \\ &= p_\infty - p_\infty = 0. \end{aligned}$$

This obviously contradicts (1).

It is well known that Lyapunov theory is a main tool for analyzing stability of systems and convergence of parameters. Barbălat Lemma in the following formulation has been extensively applied in the fields of adaptive control, parameter estimation and absolute stability, due to it is easy to establish a direct connection between Barbălat Lemma and Lyapunov theory.

Lemma 4: [9] Suppose $x : [0, \infty) \to \mathbf{R}$ is square integrable, i.e., $\lim_{t\to\infty} \int_0^t x^2(\tau)\mathrm{d}\tau < \infty$. If $\dot{x}(t)$ exists and is bounded on $[0, \infty)$, then $\lim_{t\to\infty} x(t) = 0$.

Proof. By the existence and boundedness of $\dot{x}(t)$ on $[0, \infty)$, it is easy to know that x is uniformly continuous. Therefore, from Lemma 3, we can obtain $\lim_{t\to\infty} x(t) = 0$.

In the following, we provide another proof method. Due to the existence and boundedness of $\dot{x}(t)$ on $[0, \infty)$, there exists $c > 0$ such that $|\dot{x}(t)| \leq c, \forall t \in [0, \infty)$. From this, it follows that for any $t \geq 0, \varepsilon \geq 0$, there is

$$\begin{aligned} |x^3(t) - x^3(t+\varepsilon)| &= \Big| \int_t^{t+\varepsilon} 3x^2(\tau)\dot{x}(\tau)\mathrm{d}\tau \Big| \\ &\leq \int_t^{t+\varepsilon} 3|x^2(\tau)\dot{x}(\tau)|\mathrm{d}\tau \leq \int_t^{t+\varepsilon} 3cx^2(\tau)\mathrm{d}\tau. \end{aligned} \tag{2}$$

By the square integrability of x, we know for any $\varepsilon > 0$, there exists $T > 0$ such that $(t \geq T)$

$$\int_t^{t+\varepsilon} x^2(\tau)\mathrm{d}\tau \leq \frac{\varepsilon}{3c}.$$

Then, it follows from (2) that $|x^3(t) - x^3(t+\varepsilon)| \leq \varepsilon$ for any $t \geq T$. This means that there is a constant L such that $\lim_{t\to\infty} x^3(t) = L$, i.e., $\lim_{t\to\infty} x(t) = L^{\frac{1}{3}}$. Due to $\lim_{t\to\infty} \int_0^t |x(\tau)|^2\mathrm{d}\tau < \infty$, we have L must be 0. Thus $\lim_{t\to\infty} x(t) = 0$.

Lemma 4 can not only be used to combine with Lyapunov theory to study the asymptotic convergence of systems [6], [8], but also to investigate the stability of nonlinear control systems with square-integrable disturbances [3], [11], [12]. Note that the disturbance of the system sometimes is $L_p, p \in [1, \infty)$. We thus extend Lemma 4 to Lemma 5 which has a wider range of applications. For instance, Lemma 5 can be used to investigate the L_p stability of system with L_p-disturbance, $p \in [1, \infty)$.

Lemma 5: Suppose $x : [0, \infty) \to \mathbf{R}$ is $L_p, p \in [1, \infty)$ and $\dot{x}(t)$ is bounded on $[0, \infty)$. Then $\lim_{t\to\infty} x(t) = 0$.

Proof. The proof is similar to that of Lemma 3.

The two formulations of Barbălat Lemma presented below have milder conditions, compared with various formulations of Barbălat Lemma previously mentioned. Thus they have a broader range of applications.

Lemma 6: Suppose $x : [0, \infty) \to \mathbf{R}$ is absolutely continuous. If $x(t) \in L_p, p \in [1, \infty)$ and for any compact set $C \in [0, \infty)$, $\dot{x}(t)$ is uniformly locally integrable on $[0, \infty)$, then $\lim_{t\to\infty} x(t) = 0$.

Remark 1: If for given $\varepsilon > 0$, there exists $\sigma > 0$ such that for any finite sequence of mutually independent intervals $\{(x_i, x_i')\}$ and $\sum_{i=1}^n |x_i' - x_i| < \sigma$, there is $\sum_{i=1}^n |f(x_i') - f(x_i)| < \varepsilon$, then function $f : [a, b] \to \mathbf{R}$ is absolutely continuous.

Remark 2: Let $x : [0, \infty) \to \mathbf{R}$ be a measurable function. If for any $\varepsilon > 0$, there exists $\sigma > 0$ such that for all $t \geq 0$, $\int_t^{t+\sigma} |x(t)|\mathrm{d}t \leq \varepsilon$, then x is uniformly locally integrable on a compact set C (any function belonging to $L_p, p \in [1, \infty)$ is uniformly locally integrable).

*$L_p := \Big\{x \,|\, x : [0, \infty) \to \mathbf{R}, \text{and } \big(\int_0^\infty |x(t)|^p \mathrm{d}t\big)^{1/p} < \infty\Big\}, p \in [1, \infty)$.

Proof. From the uniform local integrability of $\dot{x}$ over any closed intervals belonging to $[0,\infty)$, it follows that for given $\varepsilon > 0$, there exists $\delta > 0$ such that for $t_1 \geq 0, t_2 \leq t_1 + \delta$,

$$\Big|\int_{t_1}^{t_2} \dot{x}(\tau)\mathrm{d}\tau\Big| = |x(t_2) - x(t_1)| \leq \varepsilon.$$

From the definition of uniform continuity, it can be obtained that $x(t)$ is uniformly continuous on $[0,\infty)$. Then, by Lemma 3, we have $\lim\limits_{t\to\infty} x(t) = 0$.

We present the following lemma which is more general than Lemma 6.

Lemma 7: Suppose $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous and nondecreasing, and $\alpha(0) = 0$ if and only if $x = 0$. If $x : [0,\infty) \to \mathbf{R}$ is uniformly continuous and $\alpha(|x(t)|) \in L_1$, then $\lim\limits_{t\to\infty} \alpha(|x(t)|) = 0$ and in turn $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. Suppose by contradiction that $\lim\limits_{t\to\infty} \alpha(|x(t)|) = 0$ does not hold. Then there is a constant $\varepsilon_0 > 0$ such that for any $T > 0$, we can find $t_T > T$ to ensure $\alpha(|x(t)|) \geq \varepsilon_0$, i.e., $|x(t_T)| \geq \alpha^{-1}(\varepsilon_0)$. Based on this, we can get an infinite time sequence $\Omega = \{t_i, i = 1, 2, \cdots\}$ such that $\alpha(|x(t_i)|) \geq \varepsilon_0$, i.e., $|x(t_i)| \geq \alpha^{-1}(\varepsilon_0), \forall t_i \in \Omega$. From the uniform continuity of $x(t)$, it follows that for given ε_0, there exists $\eta(\varepsilon_0) > 0$ such that for $t', t'' \in \{t', t'' : |t' - t''| \leq \eta, t', t'' \in [0,\infty)\}$,

$$|x(t') - x(t'')| \leq \frac{\alpha^{-1}(\varepsilon_0)}{2}.$$

Therefore, for any $t \in B_\eta \triangleq \{t : |t - t_i| \leq \eta, t \in [0,\infty), t_i \in \Omega\}$, there is

$$\begin{aligned} |x(t)| &= |x(t_i) + x(t) - x(t_i)| \\ &\geq |x(t_i)| - |x(t) - x(t_i)| \geq \frac{\alpha^{-1}(\varepsilon_0)}{2}. \end{aligned}$$

The continuity of $x(t)$ implies that $x(t)$ always keeps positive or negative on Ω_q. Then, from the monotonicity and continuity of α and $\alpha(x) \neq 0$, $x \neq 0$, it follows that $\alpha(|x(t)|)$ is always positive on Ω_η. Thus, for $t_i \in \Omega$, there is

$$\begin{aligned} &\Big|\int_0^{t_i+\eta} \alpha(|x(t)|)\mathrm{d}t - \int_0^{t_i-\eta} \alpha(|x(t)|)\mathrm{d}t\Big| \\ &= \Big|\int_{t_i-\eta}^{t_i+\eta} \alpha(|x(t)|)\mathrm{d}t\Big| \geq \int_{t_i-\eta}^{t_i+\eta} \alpha(\alpha^{-1}(\varepsilon_0)/2)\mathrm{d}t \\ &\geq 2\eta\alpha(\alpha^{-1}(\varepsilon_0)/2). \end{aligned}$$

This means that

$$\lim_{t_i\to\infty, t_i\in\Omega} \int_{t_i-\eta}^{t_i+\eta} \alpha(|x(t)|)\mathrm{d}t \geq 2\eta\alpha(\alpha^{-1}(\varepsilon_0)/2). \tag{3}$$

However, it is known that $\int_0^\infty \alpha(|x(t)|)\mathrm{d}t = \alpha_\infty < \infty$. Therefore,

$$\begin{aligned} &\lim_{t_i\to\infty, t_i\in\Omega} \int_{t_i-\eta}^{t_i+\eta} \alpha(|x(t)|)\mathrm{d}t \\ &= \lim_{t_i\to\infty, t_i\in\Omega} \int_0^{t_i+\eta} \alpha(|x(t)|)\mathrm{d}t \\ &\quad - \lim_{t_i\to\infty, t_i\in\Omega} \int_0^{t_i-\eta} \alpha(|x(t)|)\mathrm{d}t \\ &= \alpha_\infty - \alpha_\infty = 0. \end{aligned}$$

This obviously contradicts (3).

C. Barbălat Lemma and Lyapunov Theory

We have already known that the Lyapunov direct method is the most commonly used method for analyzing the stability of autonomous systems (time-invariant systems). Barbărat Lemma is a commonly used method for analyzing the stability of nonautonomous systems (time-varying systems). Lyapunov theory can also be used to analyze the stability of non autonomous systems, but the shortcoming is that the required conditions are more complex and strict.

The following theorem is common Lyapunov theory used for stability analysis of nonautonomous systems.

Theorem 1: [6] Let $x = 0$ be an equilibrium point and $\Omega_R \in \mathbf{R}^n$ be a ball containing $x = 0$. There is a continuously differentiable function $V : \Omega_R \times [0,\infty) \to \mathbf{R}^+$ such that

(1) $V(x,t)$ is positive definite, that is, $V(x,t) \geq V_0(x), \forall x \in \Omega_R, \forall t \in [0,\infty)$, where $V_0 : \Omega_R \to \mathbf{R}^+$ is a positive definite function.
(2) $\dot{V}(x,t)$ is negative semidefinite, that is, $\dot{V}(x,t) \leq 0, \forall x \in \Omega_R, \forall t \in [0,\infty)$.

Then equilibrium point 0 is stable in the sense of Lyapunov on $\Omega_R \in \mathbf{R}^n$.

Moreover, if

(3) $V(x,t) \leq V_1(x), \forall x \in \Omega_R, \forall t \in [0,\infty)$, where $V_1 : \Omega_R \to \mathbf{R}^+$ is a positive definite function and $V(0,t) = 0, \forall t \in [0,\infty)$.

Then equilibrium point 0 is uniformly stable on $\Omega_R \in \mathbf{R}^n$.

If condition (2) becomes:

(2)' $\dot{V}(x,t)$ is negative definite, that is $\dot{V}(x,t) < 0, \forall x \in \Omega_R\backslash\{0\}, \forall t \in [0,\infty)$, and $\dot{V}(0,t) = 0, \forall t \in [0,\infty)$.

Then equilibrium point 0 is uniformly asymptotically stable on $\Omega_R \in \mathbf{R}^n$.

If $\Omega_R = \mathbf{R}^n$, then it is the global Lyapunov stability theory.

Although Lyapunov stability theorem mentioned above has been widely applied in stability analysis and theoretical research of practical systems, it is sometimes difficult to find Lyapunov functions with negative definite derivatives when applying this theorem to analyze asymptotic stability of systems. LaSalle invariance principle can handle situations where derivatives of Lyapunov functions are negative semidefinite, but it only applicable to autonomous systems. Barbălat Lemma makes up for the deficiencies of Lyapunov stability theorem and LaSalle invariance principle, playing a critical role in analyzing the stability of nonautonomous systems.

The following formulation of Barbălat lemma is common in stability analysis of nonautonomous systems.

Lemma 8: [6] If a continuously differentiable function $V : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ has a lower bound and $\dot{V}(x,t)$ is negative semidefinite and uniformly continuous in t, then $\lim\limits_{t\to\infty} \dot{V}(x,t) = 0$.

Remark 3: The differences between Barbălat Lemma 8 and Lyapunov stability theorem 1 lies in: (1) Lemma 8 only require $V(x,t)$ to be a function with a lower bound and not necessarily to be positive definite function; (2) In addition to ensuring that $\dot{V}(x,t)$ is negative semidefinite, Lemma 8 also requires the uniform continuity of $\dot{V}(x,t)$ with respect to t.

III. Applications of Barbălat Lemma in Analysis of System Stability

A. Asymptotic Stability Analysis

Example 1: Consider the following second-order system and analyze its stability:

$$\begin{cases} \dot{x}_1 = -x_1 + x_2\omega(t), \\ \dot{x}_2 = -x_1\omega(t). \end{cases}$$

where ω is a bounded continuous function.

Analyze. Choose $V(t) = x_1^2 + x_2^2$. Then

$$\dot{V} = 2x_1(-x_1 + x_2\omega(t)) + 2x_2(-x_1\omega(t)) = -2x_1^2 \leq 0.$$

From this, it can be obtained that $\sup_{t\geq 0} V(t) \leq V(0)$, i.e., $V(t)$ is bounded, which means x_1 and x_2 are bounded. Due to $\ddot{V}(t) = -4x_1(-x_1+x_2\omega)$ and the boundedness of $\omega(t)$, $\ddot{V}(t)$ is bounded. Thus $\dot{V}(t)$ is uniformly continuous with respect to t. It can be obtained that $\lim_{t\to\infty} \dot{V}(t) = 0$ by applying Lemma 8. Furthermore, $\lim_{t\to\infty} x_1(t) = 0$.

It should be pointed out that although x_1 eventually converges to 0, the system is not asymptotically stable. Because only the boundedness of x_2 can be guaranteed but not the asymptotic convergence of x_2.

B. Applications in Adaptive Control Design

Generally, adaptive control design is divided into the following three steps: (1) choose appropriate unknown parameters; (2) design adaptive laws for unknown parameters and adaptive controllers; (3) analyze the stability of closed-loop control systems.

The following example fully demonstrates the three steps of adaptive control design, and applies Barbălat Lemma to analyze the stability and convergence of the designed adaptive control system.

Example 2: Consider first-order nonlinear system: $\dot{x} = au + bx^2$, where $x \in \mathbf{R}$ and $u \in \mathbf{R}$ are the system state and control input, respectively; $a \neq 0$ is a constant with known sign; b is an unknown constant. Our objective is to design an adaptive controller and determine the stability and convergence of the closed-loop system.

Analyze. Choose $\theta_1 = 1/a, \theta_2 = b$. Denote the estimate of $\theta = [1/a, b]^T \in \mathbf{R}^2$ by $\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2]^T \in \mathbf{R}^2$. We use $\tilde{\theta} = [\tilde{\theta}_1, \tilde{\theta}_2]^T = \theta - \hat{\theta}$ to denote the estimate error. Since

$$\begin{aligned} b &= a \cdot \frac{1}{a} \cdot b = a \cdot \theta_1 \cdot \theta_2 \\ &= a \cdot (\hat{\theta}_1 + \tilde{\theta}_1)(\hat{\theta}_2 + \tilde{\theta}_2) \\ &= a\hat{\theta}_1\hat{\theta}_2 + a\tilde{\theta}_1\hat{\theta}_2 + a\theta_1\tilde{\theta}_2 \\ &= a\hat{\theta}_1\hat{\theta}_2 + a\tilde{\theta}_1\hat{\theta}_2 + \tilde{\theta}_2, \end{aligned}$$

system $\dot{x} = au + bx^2$ is equivalent to

$$\dot{x} = a(u + \hat{\theta}_1\hat{\theta}_2 x^2) + a\tilde{\theta}_1\hat{\theta}_2 x^2 + \tilde{\theta}_2 x^2. \tag{4}$$

Choose Lyapunov function candidate $V(x, \tilde{\theta}) = \frac{x^2}{2} + \frac{|a|}{2}\tilde{\theta}_1^2 + \frac{1}{2}\tilde{\theta}_2^2$. Note that $\dot{\tilde{\theta}}_1 = -\dot{\hat{\theta}}_1$ and $\dot{\tilde{\theta}}_2 = -\dot{\hat{\theta}}_2$. Then, taking the time derivative of V along all possible solutions of system (4), and by adding and subtracting term $-c_1x^2$ ($c_1 > 0$), we get

$$\begin{aligned} \dot{V} &= x\dot{x} + |a|\tilde{\theta}_1\dot{\tilde{\theta}}_1 + \tilde{\theta}_2\dot{\tilde{\theta}}_2 \\ &= x(a(u + \hat{\theta}_1\hat{\theta}_2x^2) + a\tilde{\theta}_1\hat{\theta}_2x^2 + \tilde{\theta}_2x^2) - |a|\tilde{\theta}_1\dot{\hat{\theta}}_1 - \tilde{\theta}_2\dot{\hat{\theta}}_2 \\ &= -c_1x^2 + a(u + c_1\hat{\theta}_1x + \hat{\theta}_1\hat{\theta}_2x^2)x - a(\dot{\hat{\theta}}_1\mathbf{sign}(a) \\ &\quad -c_1x^2 - \hat{\theta}_2x^3)\tilde{\theta}_1 - \tilde{\theta}_2(\dot{\hat{\theta}}_2 - x^3). \end{aligned} \tag{5}$$

Thus, we can design the adaptive controller as follows:

$$u = -c_1\hat{\theta}_1x - \hat{\theta}_1\hat{\theta}_2x^2, \tag{6}$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ satisfy the following adaptive regular laws:

$$\begin{cases} \dot{\hat{\theta}}_1 = \mathbf{sign}(a)(c_1 + \hat{\theta}_2x)x^2, \\ \dot{\hat{\theta}}_2 = x^3. \end{cases} \tag{7}$$

Substituting (6) and (7) into (5), we have

$$\dot{V} = -c_1x^2. \tag{8}$$

Thus V is bounded and x is square integrable, which means $x, \hat{\theta}_1$ and $\hat{\theta}_2$ are bounded. Since θ_1 and θ_2 are both constants, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are also bounded. Therefore, $\dot{x}$ is bounded. So far, the asymptotic convergence of the state has been obtained by Barbălat Lemma 4, i.e., $\lim_{t\to\infty} x(t) = 0$.

Remark 4: It is known from Example 1 and Example 2 that when V is positive definite and $\dot{V}$ is negative semidefinite. Using Lyapunov theory, we can only obtain bounded stability of the systems, but cannot ensure their asymptotic stability. Barbălat Lemma makes up for the deficiency of Lyapunov theory and is an important tool to determine the asymptotic stability of adaptive systems.

C. Stability Analysis of Systems with L_p-disturbance

Example 3: Consider the following first-order time-varying nonlinear system

$$\begin{cases} \dot{x} = a(x,t) + b(x,t)d(t), \\ y = h(x(t)). \end{cases} \tag{9}$$

where $x \in \mathbf{R}$, disturbance $d \in L_p, p \in [1, \infty), a : \mathbf{R} \times [0, \infty) \to \mathbf{R}, b : \mathbf{R} \times [0, \infty) \to \mathbf{R}, h : \mathbf{R} \to \mathbf{R}^+$. Assume that for given $t \geq 0$, a and b are continuous with respect to x, for every given x, a and b are measurable with respect to t, and for any compact set $C \in \mathbf{R}$, a and b are both uniformly bounded on $C \times \mathbf{R}^+$. h is continuous and nondecreasing. If $V : \mathbf{R} \to \mathbf{R}^+$ is locally Lipschitz, positive definite and radially unbounded, and satisfies

$$\dot{V}(x,t) \leq -h(x(t)) + |d(t)|^p, \tag{10}$$

then $\lim_{t\to\infty} h(x(t)) = 0$. If $h(x) = 0 \Leftrightarrow x = 0$, then $\lim_{t\to\infty} x(t) = 0$.

Analyze. From properties of functions a and b, the solution x of system (9) exists, and for any initial state x_0, x is absolutely continuous on $[0, \infty)$. Due to $d(t) \in L_p, p \in [1, \infty)$, $d(t)$ is uniformly locally integrable. From this and (9), $\dot{x}$ is uniformly locally integrable. Since x is absolutely continuous, x is uniformly continuous on $[0, \infty)$. From (10) and $d(t) \in L_p$, it can be obtained that $x(t)$ is bounded, i.e., $x \in L_\infty$, and $h(x)$ is integrable and its integral is bounded, i.e., $h(x) \in L_1$. Thus, we can get $\lim_{t\to\infty} h(x(t)) = 0$ by applying Barbălat Lemma 7. Furthermore, if $h(x) = 0 \Leftrightarrow x = 0$, then $\lim_{t\to\infty} x(t) = 0$.

In Example 3, if Lyapunov theory is used to analyze such nonlinear nonautonomous systems with disturbance $d(t) \in L_p$, the asymptotic convergence of the system output cannot be obtained.

IV. Conclusion

A set of primary formulations of Barbălat Lemma and its simple alternatives have been summarized. The relationships among those formulations and their applicable scopes have been investigated. First, the primary formulations of Barbălat Lemma (Lemma 1, Lemma 2 and Corollary 1) have been given. Specifically, Lemma 1 is the purely mathematical expression. Lemma 2 is obtained by replacing the uniform continuity of the function in Lemma 1 with the boundedness of its derivative. Corollary 1 is another formulation of Lemma 1. Second, some alternatives (Lemma 3-7) of Barbălat Lemma have been given. By replacing the integrability in Corollary 1 with L_p-integrability ($p \in [1, \infty)$), Lemma 3 has been proposed. Moreover, by replacing the uniform continuity of the function by the boundedness of its derivative, we have deduced Lemma 4 and Lemma 5. Under milder conditions than Lemma 1-5, Lemma 6 and Lemma 7 with a broader range of applications have been derived. Finally, the differences and connections between Barbălat Lemma and Lyapunov theory have been investigated, and a Lyapunov-like Barbălat Lemma 8 has been given. This lemma has been widely applied in the asymptotic stability theory of nonlinear time-varying systems. In order to better understand the important role of Barbălat lemma in stability analysis of systems, three examples have been given to illustrate the applications of Barbălat Lemma in analyzing asymptotic convergence of the system, adaptive control design and L_p stability.

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Barbalat引理及其在系统稳定性分析中的应用

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摘要:概述了Barbalat引理最常见的几种基本形式及其变形形式,研究了该引理各种形式之间的相互关系,并给出了各自的适用范围.通过3个例子讨论了Barbalat引理在分析系统的渐近收敛性、自适应控制设计和L_p稳定中的应用.

关键词:Barbalat引理;非线性系统;Lyapunov理论;渐近收敛性
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Barbalat Lemma and its application in analysis of system stability

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Abstract: A set of primary formulations of Barbalat Lemma and its simple alternatives are summarized. The relationships among those formulations and their applicable scopes are investigated. The applications of Barbalat Lemma in analyzing asymptotically convergence of the system, adaptive control design and L_p stability are discussed via three examples.

Key words: Barbalat Lemma; nonlinear systems; Lyapunov theory; asymptotic convergence

0 引言

众所周知,稳定性是各类控制系统最主要的性能指标,因此,发展能够有效地分析和判定控制系统稳定性的方法一直是控制理论研究的热点问题之一[1~8].19世纪末期,俄国数学家A.M.Lyapunov提出了著名的Lyapunov理论.该理论随着控制理论的产生而成为分析和研究控制系统稳定性的主要方法,并在上世纪中后期逐渐发展成熟[1~4].该方法无需求得系统的精确解,而是通过对一类似于能量函数的定性分析来判断系统的稳定性,迄今仍然是研究和判断系统,特别是非线性系统稳定性的最一般、最有效的方法.然而,经典Lyapunov稳定性理论也存在一些局限性,例如,难以构造使得其导数(半)负定的Lyapunov函数.还如,Lyapunov函数的导数半负定时,不能得到渐近稳定性结论,这有时不能满足某些实际要求.LaSalle不变集原理克服了经典Lyapunov理论的某些局限性,使得在类Lyapunov函数的导数为半负定情况下,也能得到非线性自治(时不变)系统渐近收敛性的结论[2].但是,LaSalle不变集原理不适用于非线性非自治(时变)系统渐近收敛性的分析. Barbalat引理弥补了不变集原理的不足[6].该引理虽然是关于函数及其导数渐近性的纯粹数学结论,但如果能够恰当应用,例如,能够找到导数半负定的类Lyapunov函数,那么对于非线性非自治系统,就可以得到满意渐近收敛性结论.如今,Barbalat引理在控制理论,特别是自适应控制理论起着越来越重要的作用[7]. Barbalat引理有其基本的纯粹数学表达,但随着控制理论的不断发展而变得更加丰

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富和多样,以便能更好地分析和研究各类非线性非自治系统的渐近收敛性[6,9,10].

1 Barbalat 引理

在本节中所给出的 Barbalat 引理虽然考虑的是单变量,但同样适用于多变量的情况.

1.1 Barbalat 引理的基本形式

Barbalat 引理自其被提出以来,在控制系统的稳定性分析中得到了广泛的应用,该引理最常用的表达形式如下:

引理 1[6] 设 $x:[0,\infty)\to\mathbf{R}$ 为一阶连续可导,且当 $t\to\infty$ 时有极限,则如果 $\dot{x}(t),t\in[0,\infty)$ 一致连续,那么 $\lim\limits_{t\to\infty}\dot{x}(t)=0$.

如果 $\ddot{x}(t)$ 存在且有界,那么引理1中 $\dot{x}(t)$ 的一致连续性条件可用 $\ddot{x}(t)$ 的有界性来替代,从而得到如下形式的引理.

引理 2[6] 设 $x:[0,\infty)\to\mathbf{R}$ 一阶连续可导,且当 $t\to\infty$ 时有极限,则如果 $\ddot{x}(t),t\in[0,\infty)$ 存在且有界,那么 $\lim\limits_{t\to\infty}\dot{x}(t)=0$.

如下推论是显而易见的.

推论 1 若 $x:[0,\infty)\to\mathbf{R}$ 一致连续,并且 $\lim\limits_{t\to\infty}\int_0^t x(\tau)\mathrm{d}\tau$ 存在且有界,那么 $\lim\limits_{t\to\infty}x(t)=0$.

1.2 Barbalat 引理的几种变形形式

Barbalat 引理的基本形式虽然在一定程度上能判断系统的渐近收敛性,但由于不易与 Lyapunov 理论相结合,故在实际应用中具有一定局限性.为此,对 Barbalat 基本形式进行延展和变形,得到如下几种 Barbalat 引理的表达形式.

引理 3 若 $x:[0,\infty)\to\mathbf{R}$ 一致连续,且存在 $p\in[1,\infty)$,使得 $x\in L_p$,那么 $\lim\limits_{t\to\infty}x(t)=0$.

注 $L_p:=\left\{x\mid x:[0,\infty)\to\mathbf{R},\text{且}\left(\int_0^\infty |x(t)|^p\mathrm{d}t\right)^{1/p}<\infty\right\},\ p\in[1,\infty)$.

证明 当 $p=1$ 时,因 $||x_2|-|x_1||\le|x_2-x_1|$,则由 x 的一致连续性可知 $|x|$ 亦为一致连续的.令 $F(t)=\int_0^t|x(\tau)|\mathrm{d}\tau,t\ge0$.则应用引理1易证 $\dot{F}(t)=|x(t)|$,进而得到 $x(t)$ 的收敛性.

当 $p>1$ 时,用反证法证之.假设 $\lim\limits_{t\to\infty}x(t)=0$ 不成立,那么存在常数 $\varepsilon_0>0$,对任意 $T>0$,存在 $t_T>0$,有 $|x(t_T)|\ge\varepsilon_0$.基于此,可以得到无限时间序列 $\Omega=\{t_i,i=1,2,\cdots\}$,使 $|x(t_i)|\ge\varepsilon_0,\forall t_i\in\Omega$.因为 $x(t)$ 是一致连续的,故对给定的 ε_0,存在 $\eta(\varepsilon_0)>0$,使得对任意的 $\{t',t'':|t'-t''|\le\eta,t',t''\in[0,\infty)\}$ 都有如下关系式:

$$|x(t')-x(t'')|\le\frac{\varepsilon_0}{2}.$$

由此知,对任意 $t\in B_\eta\triangleq\{t:|t-t_i|\le\eta,t\in[0,\infty),t_i\in\Omega\}$,都有

$$|x(t)|=|x(t_i)+x(t)-x(t_i)|\ge |x(t_i)|-|x(t)-x(t_i)|\ge\frac{\varepsilon_0}{2},$$

即

$$|x(t)|^p\ge\frac{\varepsilon_0^p}{2^p},\forall t\in B_\eta.$$

由 $x(t)$ 的连续性可知在域 B_η 内,$x(t)$ 恒为正或恒为负,所以,对所有的 $t_i\in\Omega$,有

$$\left|\int_0^{t_i+\eta}|x(t)|^p\mathrm{d}t-\int_0^{t_i-\eta}|x(t)|^p\mathrm{d}t\right|=\int_{t_i-\eta}^{t_i+\eta}|x(t)|^p\mathrm{d}t\ge\int_{t_i-\eta}^{t_i+\eta}\frac{\varepsilon_0^p}{2^p}\mathrm{d}t\ge2\eta\frac{\varepsilon_0^p}{2^p}.$$

这意味着

$$\lim_{t_i\to\infty,t_i\in\Omega}\int_{t_i-\eta}^{t_i+\eta}|x(t)|^p\mathrm{d}t\ge2\eta\frac{\varepsilon_0^p}{2^p}.\qquad(1)$$

而已知当 $t\to\infty$ 时,$\int_0^t|x(t)|^p\mathrm{d}t$ 存在极限,记为 p_∞,即 $\int_0^\infty|x(t)|^p\mathrm{d}t=p_\infty<\infty$,故

$$\lim_{t_i\to\infty,t_i\in\Omega}\int_{t_i-\eta}^{t_i+\eta}|x(t)|^p\mathrm{d}t=\lim_{t_i\to\infty,t_i\in\Omega}\int_0^{t_i+\eta}|x(t)|^p\mathrm{d}t-\lim_{t_i\to\infty,t_i\in\Omega}\int_0^{t_i-\eta}|x(t)|^p\mathrm{d}t=p_\infty-p_\infty=0.$$

这与式(1)相矛盾.

众所周知,Lyapunov 理论是分析系统稳定性和参数收敛性的主要工具.如下形式的 Barbalat 引理因易与 Lyapunov 理论建立直接的联系,故在自适应控制和参数估计、绝对稳定等相关领域中得到了较为广泛的应用.

引理 4[9] 设 $x:[0,\infty)\to\mathbf{R}$ 平方可积,即 $\lim\limits_{t\to\infty}\int_0^t x^2(\tau)\mathrm{d}\tau<\infty$.则如果 $\dot{x}(t),t\in[0,\infty)$,存在且有界,那么 $\lim\limits_{t\to\infty}x(t)=0$.

证明 由 $\dot{x}(t),t\in[0,\infty)$ 存在且有界,易知 x 是一致连续的,所以由引理3可得 $\lim\limits_{t\to\infty}x(t)=0$.

下面给出另外一种证明方法.因为 $\dot{x}(t),t\in[0,\infty)$ 存在且有界,故存在 $c>0$,使得 $|\dot{x}(t)|\le c,\forall t\in[0,\infty)$.由此可知,对任意 $t\ge0,\varepsilon\ge0$,有

$$|x^3(t)-x^3(t+\varepsilon)|=\left|\int_t^{t+\varepsilon}3x^2(\tau)\dot{x}(\tau)\mathrm{d}\tau\right|\le$$

$$\int_t^{t+\varepsilon}3|x^2(\tau)\dot{x}(\tau)|\mathrm{d}\tau\le\int_t^{t+\varepsilon}3cx^2(\tau)\mathrm{d}\tau.\quad(2)$$

进而，由 x 的平方可积性可知，对任意的 $\varepsilon>0$，存在 $T>0$，当 $t\ge T$ 时，有

$$\int_t^{t+\varepsilon}x^2(\tau)\mathrm{d}\tau\le\frac{\varepsilon}{3c}.$$

由此及式(2) 可知，对任意 $t\ge T$，$|x^3(t)-x^3(t+\varepsilon)|\le\varepsilon$. 这意味着存在常数 L，使得 $\lim\limits_{t\to\infty}x^3(t)=L$，即 $\lim\limits_{t\to\infty}x(t)=L^{1/3}$. 又因为 $\lim\limits_{t\to\infty}\int_0^t|x(\tau)|^2\mathrm{d}\tau<\infty$，所以 L 必为 0，故有 $\lim\limits_{t\to\infty}x(t)=0$.

引理 4 不仅可以用来与 Lyapunov 理论相结合研究系统的渐近收敛性[6,8]，还可以用来研究具有平方可积性扰动的非线性控制系统的稳定性[3,11,12]. 但有时系统的扰动是 $L_p,p\in[1,\infty)$，如此，可将引理 4 推广到如下适用范围更为广泛的引理 5，例如，可用于研究具有 $L_p,p\in[1,\infty)$ 扰动系统 L_p 的稳定性.

引理 5　设 $x:[0,\infty)\to\mathbf{R}$ 为 $L_p,p\in[1,\infty)$ 的，且 $\dot{x}(t),t\in[0,\infty)$ 有界，那么 $\lim\limits_{t\to\infty}x(t)=0$.

证明　参照引理 3 的证明过程易证.

如下所给出的 Barbalat 引理的两种形式较前述各种形式的 Barbalat 引理所成立的条件更弱，因而具有更为广泛的适用范围.

引理 6　设 $x:[0,\infty)\to\mathbf{R}$ 绝对连续，则如果 $x(t)\in L_p,p\in[1,\infty)$，且 $\dot{x}(t),t\in[0,\infty)$ 对任意紧集 $C\in[0,\infty)$ 一致局部可积，那么 $\lim\limits_{t\to\infty}x(t)=0$.

注 1　如果对给定 $\varepsilon>0$，存在 $\sigma>0$，使得对任意有限的相互独立区间序 $\{(x_i,x'_i)\}$，且 $\sum\limits_{i=1}^{n}|x'_i-x_i|<\sigma$，总有 $\sum\limits_{i=1}^{n}|f(x'_i)-f(x_i)|<\varepsilon$，则称函数 $f:[a,b]\to\mathbf{R}$ 为绝对连续的.

注 2　设 $x:[0,\infty)\to\mathbf{R}$ 为可测函数. 若对任意 $\varepsilon>0$，存在 $\sigma>0$，使得对所有 $t\ge0$，都有 $\int_t^{t+\sigma}|x(t)|\mathrm{d}t\le\varepsilon$，则称 x 在紧集 C 上一致局部可积（任意属于 $L_p,p\in[1,\infty)$ 的函数皆为一致局部可积）.

证明　对给定的 $\varepsilon>0$，由 $\dot{x}$ 在属于 $[0,\infty)$ 的任意闭区间上的一致局部可积性知，存在 $\delta>0$ 对任意 $t_1\ge0,t_2\le t_1+\delta$ 都有

$$\left|\int_{t_1}^{t_2}\dot{x}(\tau)\mathrm{d}\tau\right|=|x(t_2)-x(t_1)|\le\varepsilon,$$

则由一致连续性的定义易知 $x(t)$ 在 $[0,\infty)$ 上是一致连续的，由此及引理 3 可得到 $\lim\limits_{t\to\infty}x(t)=0$.

较引理 6 更为一般的是如下引理 7.

引理 7　设 $\alpha:\mathbf{R}^+\to\mathbf{R}^+$ 连续、非减，且仅当 $x=0$ 时，$\alpha(0)=0$. 则如果 $x:[0,\infty)\to\mathbf{R}$ 为一致连续且 $\alpha(|x(t)|)\in L_1$，那么 $\lim\limits_{t\to\infty}\alpha(|x(t)|)=0$，进而 $\lim\limits_{t\to\infty}x(t)=0$.

证明　用反证法证之. 假设 $\lim\limits_{t\to\infty}\alpha(|x(t)|)=0$ 不成立，那么存在常数 $\varepsilon_0>0$，对任意 $T>0$，存在 $t_T>T$，有 $\alpha(|x(t)|)\ge\varepsilon_0$，即 $|x(t_T)|\ge\alpha^{-1}(\varepsilon_0)$. 基于此，可以得到无限时间序列 $\Omega=\{t_i,i=1,2,\cdots\}$，使得 $\alpha(|x(t_i)|)\ge\varepsilon_0$，即 $|x(t_i)|\ge\alpha^{-1}(\varepsilon_0)$，$\forall t_i\in\Omega$. 因为 $x(t)$ 是一致连续的，故对给定的 ε_0，存在 $\eta(\varepsilon_0)>0$，使得对任意的 $\{t',t'':|t'-t''|\le\eta,t',t''\in[0,\infty)\}$，都有关系式

$$|x(t')-x(t'')|\le\frac{\alpha^{-1}(\varepsilon_0)}{2}$$

成立. 由此可知对任意 $t\in B_\eta\triangleq\{t:|t-t_i|\le\eta,t\in[0,\infty),t_i\in\Omega\}$，都有

$$|x(t)|=|x(t_i)+x(t)-x(t_i)|\ge$$

$$|x(t_i)|-|x(t)-x(t_i)|\ge\frac{\alpha^{-1}(\varepsilon_0)}{2}$$

成立. 由 $x(t)$ 的连续性可知，在域 Ω_η 内 $x(t)$ 恒为正或恒为负，进而由 α 的单调性、连续性及 $\alpha(x)\ne0$，$x\ne0$ 可知，在域 Ω_η 内 $\alpha(|x(t)|)$ 恒为正. 所以，对所有的 $t_i\in\Omega$，有

$$\left|\int_0^{t_i+\eta}\alpha(|x(t)|)\mathrm{d}t-\int_0^{t_i-\eta}\alpha(|x(t)|)\mathrm{d}t\right|=$$

$$\left|\int_{t_i-\eta}^{t_i+\eta}\alpha(|x(t)|)\mathrm{d}t\right|\ge\int_{t_i-\eta}^{t_i+\eta}\alpha(\alpha^{-1}(\varepsilon_0)/2)\mathrm{d}t\ge$$

$$2\eta\alpha(\alpha^{-1}(\varepsilon_0)/2).$$

这意味着：

$$\lim_{t_i\to\infty,t_i\in\Omega}\int_{t_i-\eta}^{t_i+\eta}\alpha(|x(t)|)\mathrm{d}t\ge2\eta\alpha(\alpha^{-1}(\varepsilon_0)/2).\tag{3}$$

而已知当 $t\to\infty$ 时，$\int_0^t\alpha(|x(t)|)\mathrm{d}t$ 存在极限，记为 α_∞，即 $\int_0^\infty\alpha|x(t)|\mathrm{d}t=\alpha_\infty<\infty$，故

$$\lim_{t_i\to\infty,t_i\in\Omega}\int_{t_i-\eta}^{t_i+\eta}\alpha(|x(t)|)\mathrm{d}t=$$

$$\lim_{t_i\to\infty,t_i\in\Omega}\int_0^{t_i+\eta}\alpha(|x(t)|)\mathrm{d}t-$$

$$\lim_{t_i\to\infty,t_i\in\Omega}\int_0^{t_i-\eta}\alpha(|x(t)|)\mathrm{d}t=$$

$$\alpha_\infty - \alpha_\infty = 0.$$

这与式(3)相矛盾.

1.3 Barbalat引理与Lyapunov理论

我们已经知道,Lyapunov直接法是分析自治系统(即时不变系统)稳定性的最常用方法.Barbalat引理是分析非自治系统(即时变系统)稳定性的常用方法.Lyapunov理论也可以用于分析非自治系统的稳定性,但缺点是所需的条件更为复杂和严格.

如下定理是常用于非自治系统稳定性分析的Lyapunov理论.

定理1[6] 若在平衡点0附近的球域$\Omega_R \in \mathbf{R}^n$内,存在连续可微的Lyapunov函数$V:\Omega_R \times [0,\infty) \to \mathbf{R}^+$,使得

(1) $V(x,t)$为正定,即$V(x,t) \geq V_0(x)$, $\forall x \in \Omega_R, \forall t \in [0,\infty)$,其中$V_0:\Omega_R \to \mathbf{R}^+$是正定函数.

(2) $\dot{V}(x,t)$为半负定,即$\dot{V}(x,t) \leq 0, \forall x \in \Omega_R, \forall t \in [0,\infty)$.

则平衡点0在域$\Omega_R \in \mathbf{R}^n$内为Lyapunov意义下稳定的.此外,若

(3) $V(x,t) \leq V_1(x), \forall x \in \Omega_R, \forall t \in [0,\infty)$,其中$V_1:\Omega_R \to \mathbf{R}^+$是正定函数,且$V(0,t)=0$, $\forall t \in [0,\infty)$.

那么平衡点0在域$\Omega_R \in \mathbf{R}^n$内一致稳定.若条件(2)变为:

(2)' $\dot{V}(x,t)$为负定,即$\dot{V}(x,t) < 0, \forall x \in \Omega_R \setminus \{0\}, \forall t \in [0,\infty)$,且$\dot{V}(0,t) = 0, \forall t \in [0,\infty)$.

那么平衡点0在域$\Omega_R \in \mathbf{R}^n$内一致渐近稳定.

若域$\Omega_R \in \mathbf{R}^n$为整个实数空间$\mathbf{R}^n$,则为全局Lyapunov稳定性理论.

如上Lyapunov稳定性定理虽然在实际系统稳定性分析和理论研究中得到了广泛应用,但应用该定理分析系统的渐近稳定性时,导数为负定的Lyapunov函数有时难以找到,尽管LaSalle不变集原理可以处理Lyapunov函数的导数为半负定的情况,但是只适用于自治系统.Barbalat引理弥补了Lyapunov稳定性定理和Lasalle不变集原理的不足,在分析非自治系统稳定性方面起到了十分关键的作用.

如下是Barbalat引理在非自治系统的稳定性分析中常见的表述形式.

引理8[6] 如果连续可导的二元函数$V:\mathbf{R}^n \times [0,\infty) \to \mathbf{R}$有下界,$\dot{V}(x,t)$半负定,且$\dot{V}(x,t)$关于时间$t$是一致连续的,那么,$\lim\limits_{t\to\infty}\dot{V}(x,t) = 0$.

注1 Barbalat引理8与Lyapunov稳定性定理1的不同之处有:(1)在引理8中只要求$V(x,t)$是有下界的而不一定是正定的函数;(2)在引理8中除了要保证$\dot{V}(x,t)$是半负定以外,还要满足关于时间t一致连续性的条件.

2 Barbalat引理在系统稳定性分析中的应用

2.1 Barbalat引理用于系统渐近稳定性分析

例1 考虑如下二阶系统:

$$\dot{x}_1 = -x_1 + x_2 w(t),$$
$$\dot{x}_2 = -x_1 w(t).$$

其中w是一有界连续函数,分析系统的稳定性.

分析 选取$V(t) = x_1^2 + x_2^2$,则

$$\dot{V} = 2x_1(-x_1 + x_2 w(t)) + 2x_2(-x_1 w(t)) = -2x_1^2 \leq 0.$$

由此可得$\sup\limits_{t\geq 0} V(t) \leq V(0)$,即$V(t)$为有界的.这意味着$x_1$及$x_2$为有界的.由此及$\ddot{V}(t) = -4x_1(-x_1 + x_2 w)$和$w(t)$的有界性可知$\ddot{V}(t)$是有界的,所以$\dot{V}(t)$关于时间$t$是一致连续的.那么应用引理8可得$\lim\limits_{t\to\infty}\dot{V}(t) = 0$,进而可得$\lim\limits_{t\to\infty}x_1(t) = 0$.

需指出的是,虽然x_1最终收敛于0,但是整个系统不是渐近稳定的,因为只能保证x_2有界,而不能保证x_2的渐近收敛性.

2.2 Barbalat引理在自适应控制设计中的应用

一般地,自适应控制设计分为以下3个步骤:(1)选择适当的未知参数;(2)设计未知参数的自适应律和自适应控制器;(3)分析闭环控制系统的稳定性.

下面的例子充分体现了自适应控制设计的3个步骤,并且应用Barbalat引理分析了所设计的自适应控制系统的稳定性和收敛性.

例2 考虑一阶非线性控制系统:$\dot{x} = au + bx^2$,其中$x \in \mathbf{R}$和$u \in \mathbf{R}$分别是系统状态和控制输入,a为符号已知且不为0的常数,b为未知常数.我们的目标是设计自适应控制器并判断闭环系统的稳定性和收敛性.

分析 选取$\theta_1 = 1/a, \theta_2 = b$.记$\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2]^{\mathrm{T}} \in \mathbf{R}^2$为$\theta = [1/a, b]^{\mathrm{T}} \in \mathbf{R}^2$的估计,并记$\tilde{\theta} = [\tilde{\theta}_1, \tilde{\theta}_1]^{\mathrm{T}} = \theta - \hat{\theta}$为估计误差.因为

$$b = a \cdot \frac{1}{a} \cdot b = a \cdot \theta_1 \cdot \theta_2 =$$

$$a\cdot(\hat{\theta}_1+\tilde{\theta}_1)(\hat{\theta}_2+\tilde{\theta}_2)=$$
$$a\hat{\theta}_1\hat{\theta}_2+a\tilde{\theta}_1\hat{\theta}_2+a\theta_1\tilde{\theta}_2=$$
$$a\hat{\theta}_1\hat{\theta}_2+a\tilde{\theta}_1\hat{\theta}_2+\tilde{\theta}_2.$$

所以，系统 $\dot{x}=au+bx^2$ 等价于

$$\dot{x}=a(u+\hat{\theta}_1\hat{\theta}_2x^2)+a\tilde{\theta}_1\hat{\theta}_2x^2+\tilde{\theta}_2x^2. \quad (4)$$

选取 Lyapunov 候选函数 $V(x,\tilde{\theta})=\frac{x^2}{2}+\frac{|a|}{2}\tilde{\theta}_1^2+\frac{1}{2}\tilde{\theta}_2^2$. 注意到 $\dot{\tilde{\theta}}_1=-\dot{\hat{\theta}}_1$ 和 $\dot{\tilde{\theta}}_2=-\dot{\hat{\theta}}_2$，则沿系统(4)的所有可能解求 V 的导数并在等式右边加减项 $-c_1x^2, c_1>0$，整理后得到：

$$\dot{V}=x\dot{x}+|a|\tilde{\theta}_1\dot{\tilde{\theta}}_1+\tilde{\theta}_2\dot{\tilde{\theta}}_2=x(a(u+\hat{\theta}_1\hat{\theta}_2x^2)+$$
$$a\tilde{\theta}_1\hat{\theta}_2x^2+\tilde{\theta}_2x^2)-|a|\tilde{\theta}_1\dot{\hat{\theta}}_1-\tilde{\theta}_2\dot{\hat{\theta}}_2=$$
$$-c_1x^2+a(u+c_1\hat{\theta}_1x+\hat{\theta}_1\hat{\theta}_2x^2)x-$$
$$a(\dot{\hat{\theta}}_1\mathrm{sgn}a-c_1x^2-\hat{\theta}_2x^3)\tilde{\theta}_1-\tilde{\theta}_2(\dot{\hat{\theta}}_2-x^3). \quad (5)$$

因此，可设计自适应控制器：

$$u=-c_1\hat{\theta}_1x-\hat{\theta}_1\hat{\theta}_2x^2, \quad (6)$$

其中 $\hat{\theta}_1$ 和 $\hat{\theta}_2$ 满足如下自适应调节律：

$$\dot{\hat{\theta}}_1=\mathrm{sgn}a(c_1+\hat{\theta}_2x)x^2,\ \dot{\hat{\theta}}_2=x^3. \quad (7)$$

将式(6)，(7)带入式(5)中可得：

$$\dot{V}=-c_1x^2. \quad (8)$$

由此可知 V 是有界的且 x 为平方可积的，这意味着 $x,\hat{\theta}_1$ 和 $\hat{\theta}_2$ 都是有界的. 因为 θ_1 和 θ_2 皆为常数，从而 $\tilde{\theta}_1$ 和 $\tilde{\theta}_2$ 也是有界的，如此易知 $\dot{x}$ 为有界的. 至此，应用 Barbalat 引理 4 可得状态的渐进收敛性，即 $\lim\limits_{t\to\infty}x(t)=0$.

注 2　由例 1 及例 2 可以看出，当 V 为正定且 $\dot{V}$ 为半负定时，运用 Lyapunov 理论只能得到系统的有界稳定性，而不能确保系统的渐近稳定性. Barbalat 引理弥补了 Lyapunov 理论的不足，是判断自适应系统渐近稳定性的重要工具.

2.3　Barbalat 引理用于分析有 $L_p, p\in[1,\infty)$ 扰动系统的稳定性

例 3　考虑如下一阶时变非线性系统：

$$\begin{aligned}\dot{x}&=a(x,t)+b(x,t)d(t),\\ y&=h(x(t)).\end{aligned} \quad (9)$$

其中，$x\in\mathbf{R}$，扰动项 $d\in L_p, p\in[1,\infty)$，$a:\mathbf{R}\in[0,\infty)\to\mathbf{R}$，$b:\mathbf{R}\times[0,\infty)\to\mathbf{R}$，$h:\mathbf{R}\to\mathbf{R}^+$. 假设对每个给定的 $t\geq0$，a 和 b 均关于 x 连续，对每个给定的 x，a 和 b 均关于 t 可测，且对任意紧集 $C\in\mathbf{R}$，a 和 b 在 $C\times\mathbf{R}^+$ 上均一致有界，函数 h 为连续、非减. 如果 $V:\mathbf{R}\to\mathbf{R}^+$ 是局部 Lipschitz，正定、径向无界函数，并使得

$$\dot{V}(x,t)\leq-h(x(t))+|d(t)|^p, \quad (10)$$

则 $\lim\limits_{t\to\infty}h(x(t))=0$. 若 $h(x)=0\Leftrightarrow x=0$，则 $\lim\limits_{t\to\infty}x(t)=0$.

分析　由函数 a 和 b 的性质易知系统(9)的解 x 存在，并且对任意的初始状态 x_0，解 x 在 $[0,\infty)$ 上都是绝对连续的. 又因为 $d(t)\in L_p, p\in[1,\infty)$，所以 $d(t)$ 为一致局部可积. 由此及式(9)可知 $\dot{x}$ 为一致局部可积的. 由这以及 x 的绝对连续性可知 x 在 $[0,\infty)$ 上为一致连续的. 由(10)以及 $d(t)\in L_p$，可得 $x(t)$ 是有界的，即 $x\in L_\infty$，以及 $h(x)$ 为积分有界的，即 $h(x)\in L_1$，因此据 Barbalat 引理 7 可得 $\lim\limits_{t\to\infty}h(x(t))=0$. 进而，若 $h(x)=0\Leftrightarrow x=0$，则 $\lim\limits_{t\to\infty}x(t)=0$.

在例 3 中，若运用 Lyapunov 理论来分析此类带有扰动项 $d(t)\in L_p$ 的非线性非自治系统，将得不到系统输出渐近收敛的结论.

3　结论

概述了 Barbalat 引理最常见的几种基本形式及其延展和变形形式，研究了该引理各种形式之间的相互关系、给出了各自的适用范围. 首先给出并证明了 Barbalat 引理的基本表达形式(引理 1、引理 2 和推论 1)，其中引理 1 是 Barbalat 引理的纯粹数学表达；引理 2 是将引理 1 中的函数的一致连续性替换为该函数导数的有界性而得到的；推论 1 仅为引理 1 的另外一种表达. 其次，给出了 Barbalat 引理的几种延展和变形形式(引理 3 ~ 7). 其中，将推论 1 条件中的可积性替换为 $L_p, p\in[1,\infty)$ 可积性，从而得到引理 3；将引理 3 中的函数一致连续性替换为该函数导数的有界性，从而得到引理 4 和引理 5；在较引理 1 ~ 5 更弱条件下，得到了适用范围更广的引理 6 和引理 7. 最后，研究了 Barbalat 引理与 Lyapunov 理论的区别与联系，给出了类 Lyapunov 的 Barbalat 引理 8. 该引理在非线性时变系统的渐近稳定性理论中得到大量应用. 为了更好地认识 Barbalat 引理在系统稳定性分析中的重要作用，通过 3 个例子分别说明了 Barbalat 引理在分析系统的渐近收敛性、自适应稳定控制设计和 L_p 稳定中的应用情况.

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