Supplementary Article of Paper

"Adaptive Prescribed-Time Stabilization for Uncertain Unmeasured-State-Dependent Systems"

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For the paper (Adaptive Prescribed-Time Stabilization for Uncertain Unmeasured-State-Dependent Systems), we provide proof of Proposition 1, proof of Lemma 2, properties of auxiliary analysis states, proofs of Lemma 4 and Theorem 3 about uniform boundedness, and a detailed equivalent transformation.

This article is not self-contained and hence reading the article needs to refer to the paper.

We state that the labels, such as (10) and (22), refer to the ones in the paper, while the labels with capital letter S, such as (S.1) and (S.2), are merely used in this supplementary article.

I. Detailed Proof of Proposition 1

Proposition 1: Let $V_\varepsilon = L\varepsilon^{\mathrm{T}}P\varepsilon$ with P satisfying (10). Then along the trajectories of ε-system (21), there is

$$\frac{\mathrm{d}V_\varepsilon}{\mathrm{d}\tau} \le -\big(\tfrac{L^2}{2}-\theta_\varepsilon\big)\|\varepsilon\|^2+\tfrac{L^2}{4\nu_1}\|\zeta\|^2+\theta_\varepsilon\bar{y}^2\varphi^2(\bar{y})+\theta_\varepsilon, \quad (22)$$

where $\nu_1 = 4\|Qa\|^2$, unknown constant $\theta_\varepsilon > 0$ and known smooth function $\varphi(\cdot)$ satisfies $\varphi(\bar{y}) \ge \bar{\phi}(\bar{y}) + |\tilde{\phi}(\bar{y})|$ as before.

Proof. By $\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}$ in (21), we get

$$\begin{aligned}\frac{\mathrm{d}V_\varepsilon}{\mathrm{d}\tau} &= L^2\varepsilon^{\mathrm{T}}\big(A_a^{\mathrm{T}}P+PA_a\big)\varepsilon+2L\varepsilon^{\mathrm{T}}P\Lambda(\cdot)f(\cdot)\\ &\quad+2L\varepsilon^{\mathrm{T}}Paf_1\gamma^{n-1}+L\varepsilon^{\mathrm{T}}(P\bar{D}_{n-1}+\bar{D}_{n-1}^{\mathrm{T}}P)\varepsilon\tfrac{\gamma'(\tau)}{\gamma(\tau)}\\ &\quad-\varepsilon^{\mathrm{T}}\Big(P\big(D_{n-1}-\tfrac{\mathbf{I}_{n-1}}{2}\big)+\big(D_{n-1}-\tfrac{\mathbf{I}_{n-1}}{2}\big)^{\mathrm{T}}P\Big)\varepsilon\tfrac{\mathrm{d}L}{\mathrm{d}\tau}. \quad (\text{S.1})\end{aligned}$$

Recall from (21) and $\beta(\cdot) \ge \beta_1(M,\bar{y})+\beta_2(\bar{y})$ that $-\frac{\mathrm{d}L}{\mathrm{d}\tau} \le \delta_1L^2-\delta_2L\big(\beta_1(\cdot)+\beta_2(\cdot)\big)$. Then by (10), the last term in (S.1) satisfies (noting $\delta_1\bar{c}_1 \le \frac{1}{4}$ and $\delta_2\underline{c}_1 \ge 1$)

$$\begin{aligned}&-\varepsilon^{\mathrm{T}}\Big(P\big(D_{n-1}-\tfrac{1}{2}\mathbf{I}_{n-1}\big)+\big(D_{n-1}-\tfrac{1}{2}\mathbf{I}_{n-1}\big)^{\mathrm{T}}P\Big)\varepsilon\tfrac{\mathrm{d}L}{\mathrm{d}\tau}\\ &\le \delta_1\bar{c}_1L^2\|\varepsilon\|^2-\delta_2\underline{c}_1L\big(\beta_1(\cdot)+\beta_2(\cdot)\big)\|\varepsilon\|^2\\ &\le \tfrac{L^2}{4}\|\varepsilon\|^2-L\big(\beta_1(\cdot)+\beta_2(\cdot)\big)\|\varepsilon\|^2.\end{aligned}$$

Putting this into (S.1) and noting from (10) that $L^2\varepsilon^{\mathrm{T}}\big(A_a^{\mathrm{T}}P+PA_a\big)\varepsilon \le -2L^2\|\varepsilon\|^2$, we have

$$\begin{aligned}\frac{\mathrm{d}V_\varepsilon}{\mathrm{d}\tau} &\le -\tfrac{7}{4}L^2\|\varepsilon\|^2-L\big(\beta_1(\cdot)+\beta_2(\cdot)\big)\|\varepsilon\|^2+2L\varepsilon^{\mathrm{T}}P\Lambda(\cdot)f(\cdot)\\ &\quad+2L\varepsilon^{\mathrm{T}}Paf_1\gamma^{n-1}+L\varepsilon^{\mathrm{T}}(P\bar{D}_{n-1}+\bar{D}_{n-1}^{\mathrm{T}}P)\varepsilon\tfrac{\gamma'(\tau)}{\gamma(\tau)}. \quad (\text{S.2})\end{aligned}$$

Noticing that $\|\Lambda(\cdot)f(\cdot)\|$ satisfies (see the end of this section for the detailed estimate):

$$\|\Lambda(\cdot)f(\cdot)\| \le \tfrac{\bar{\theta}}{L}\sqrt{\beta_1(\cdot)}+\bar{\theta}\bar{\phi}(\bar{y})(1+\|\varepsilon\|+\|\zeta\|), \quad (\text{S.3})$$

and then substituting it into the 3rd term in (S.2), we obtain:

$$\begin{aligned}2L\varepsilon^{\mathrm{T}}P\Lambda(\cdot)f(\cdot) &\le 2\bar{\theta}\|P\|\sqrt{\beta_1(\cdot)}\|\varepsilon\|+2\bar{\theta}\|P\|L\bar{\phi}(\bar{y})\|\varepsilon\|\\ &\quad+2\bar{\theta}\|P\|L\bar{\phi}(\bar{y})\|\varepsilon\|(\|\varepsilon\|+\|\zeta\|). \quad (\text{S.4})\end{aligned}$$

By completing the square, the first two terms of (S.4) satisfy

$$\begin{cases}2\bar{\theta}\|P\|\sqrt{\beta_1(\cdot)}\|\varepsilon\| \le \beta_1(\cdot)\|\varepsilon\|^2+\|P\|^2\bar{\theta}^2,\\ 2\bar{\theta}\|P\|L\bar{\phi}(\bar{y})\|\varepsilon\| \le \tfrac{L^2}{4}\|\varepsilon\|^2+4\bar{\theta}^2\|P\|^2\bar{\phi}^2(\bar{y}),\end{cases} \quad (\text{S.5})$$

where "$\beta_1(\cdot)\|\varepsilon\|^2$" and "$\frac{L^2}{4}\|\varepsilon\|^2$" can be eliminated by the negative terms in (S.2) (using $L \ge 1$).

As for the 3rd term in (S.4), we similarly have

$$\begin{aligned}&2\bar{\theta}\|P\|L\bar{\phi}(\bar{y})\|\varepsilon\|(\|\varepsilon\|+\|\zeta\|)\\ &\le \tfrac{L^2}{4}\|\varepsilon\|^2+\tfrac{L^2}{4\nu_1}\|\zeta\|^2+4(1+\nu_1)\bar{\theta}^2\|P\|^2\bar{\phi}^2(\bar{y})\|\varepsilon\|^2\\ &\le \tfrac{L^2}{4}\|\varepsilon\|^2+\tfrac{L^2}{4\nu_1}\|\zeta\|^2+4(1+\nu_1)^2\bar{\theta}^4\|P\|^4\|\varepsilon\|^2+\bar{\phi}^4(\bar{y})\|\varepsilon\|^2,\end{aligned}$$

where "$\frac{L^2}{4}\|\varepsilon\|^2$" and "$\bar{\phi}^4(\bar{y})\|\varepsilon\|^2$" can also be eliminated by the negative terms in (S.2) (using $L \ge 1$ and $\beta_2 = \bar{\phi}^4(\bar{y})$).

To deal with the 4th term in (S.2), we recall from Assumption 1 that $|f_1(\cdot)| \le \theta\phi(y)\big(|y|+c|y|^{m_{1,1}}\big)$ and $n-1 \le nm_{1,1}$. Then by $y = \frac{\bar{y}}{\gamma^n(\tau)}$ and $\phi(y) \le \bar{\phi}(\bar{y})$ for a smooth positive function $\bar{\phi}(\bar{y})$, we have (using $\gamma(\tau) \ge 1$ and $c|\bar{y}|^{m_{1,1}} \le |\bar{y}|+1$)

$$\begin{aligned}|f_1(\cdot)\gamma^{n-1}(\tau)| &\le \theta\bar{\phi}(\bar{y})\big(\tfrac{|\bar{y}|}{\gamma^n(\tau)}+\tfrac{c|\bar{y}|^{m_{1,1}}}{\gamma^{nm_{1,1}}(\tau)}\big)\gamma^{n-1}(\tau)\\ &\le \theta\bar{\phi}(\bar{y})(2|\bar{y}|+1). \quad (\text{S.6})\end{aligned}$$

By this, the 4th term satisfies (using completing the square)

$$\begin{aligned}&2L\varepsilon^{\mathrm{T}}Paf_1(\cdot)\gamma^{n-1}(\tau)\\ &\le \tfrac{L^2}{2}\|\varepsilon\|^2+16\|Pa\|^2\theta^2\bar{y}^2\bar{\phi}^2(\bar{y})+4\|Pa\|^2\theta^2\bar{\phi}^2(\bar{y}). \quad (\text{S.7})\end{aligned}$$

Now we recall from (19) that $P\bar{D}_{n-1}+\bar{D}_{n-1}^{\mathrm{T}}P \le c_1\mathbf{I}_{n-1}$ for some $c_1 > 0$ and from (4) that $\frac{\gamma'(\tau)}{\gamma(\tau)} \le 1$. It is then clear (noting $L(t) \ge L(0) \ge 4c_1$ in (21))

$$L\varepsilon^{\mathrm{T}}(P\bar{D}_{n-1}+\bar{D}_{n-1}^{\mathrm{T}}P)\varepsilon\tfrac{\gamma'(\tau)}{\gamma(\tau)} \le c_1L\|\varepsilon\|^2 \le \tfrac{L^2}{4}\|\varepsilon\|^2.$$

Plug the estimates above into (S.2). For the terms "$4\bar{\theta}^2\|P\|^2\bar{\phi}^2(\bar{y})$" in (S.5) and "$4\|Pa\|^2\theta^2\bar{\phi}^2(\bar{y})$" in (S.7), employ the decomposition $\bar{\phi}(\bar{y}) = \bar{y}\tilde{\phi}(\bar{y})+\bar{\phi}(0)$ with $\tilde{\phi}(\cdot)$ a known smooth function satisfying $\varphi^2(\bar{y}) \ge \max\{\bar{\phi}^2(\bar{y}), \tilde{\phi}^2(\bar{y})\}$. After collecting some terms, we directly arrive at (22) with some constant $\theta_\varepsilon > 0$. □

Detailed estimate of $\|\Lambda(\cdot)f(\cdot)\|$: We first present the following inequalities with lower power $p \in (0,1)$:

$$|\chi|^p \le |\chi|+1, \qquad |\chi_1+\chi_2|^p \le |\chi_1|^p+|\chi_2|^p. \quad (\text{S.8})$$

Noting $\dot{\mu}(t)\big|_{t=\mu^{-1}(\tau)} = \gamma(\tau)$ in (3), we see from (8) that $x_i = \frac{\bar{x}_i}{\gamma^{n-i+1}(\tau)}+\frac{L^{i-1}\varepsilon_i}{\gamma^{n-i+1}(\tau)}$ by use of $\hat{x}_i = \frac{\bar{x}_i}{\gamma^{n-i+1}(\tau)}$ in (11).

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Then from Assumption 1 and $\phi\big(\frac{\bar{y}}{\gamma^n(\tau)}\big) \le \bar{\phi}(\bar{y})$ required above and by $y = \frac{\bar{y}}{\gamma^n(\tau)}$ and (S.8), we know

$$\begin{aligned}\|\Lambda(\cdot)f(\cdot)\| \ \le\ & \theta\bar{\phi}(\bar{y})\sum_{i=2}^{n}\frac{\gamma^{n-i}(\tau)}{L^{i-1}}\bigg(\frac{|\bar{y}|}{\gamma^n(\tau)} + \frac{|\bar{y}|^{m_{i,1}}}{\gamma^{n\times m_{i,1}}(\tau)} \\ & + \sum_{j=2}^{i}\big(\frac{L^{j-1}|\varepsilon_j|}{\gamma^{n-j+1}(\tau)} + \frac{L^{m_{i,j}\times(j-1)}|\varepsilon_j|^{m_{i,j}}}{\gamma^{m_{i,j}\times(n-j+1)}(\tau)}\big) \\ & + \sum_{j=2}^{i}\big(\frac{|\bar{x}_j|}{\gamma^{n-j+1}(\tau)} + \frac{|\bar{x}_j|^{m_{i,j}}}{\gamma^{m_{i,j}\times(n-j+1)}(\tau)}\big)\bigg). \quad \text{(S.9)}\end{aligned}$$

We next estimate the terms in (S.9). Note from Assumption 1 that $n\times m_{i,1} - (n-i) \ge 0$. Then by $\gamma(\tau)\ge 1$ (in (4)) and (S.8) with $p = m_{i,1} < 1$, we have

$$\frac{\gamma^{n-i}(\tau)}{L^{i-1}}\Big(\frac{|\bar{y}|}{\gamma^n(\tau)} + \frac{|\bar{y}|^{m_{i,1}}}{\gamma^{n\times m_{i,1}}(\tau)}\Big) \le \frac{|\bar{y}|+|\bar{y}|^{m_{i,1}}}{L^{i-1}} \le \frac{2|\bar{y}|+1}{L^{i-1}}.$$

Similarly, using $(n-j+1)\times m_{i,j} - (n-i) \ge 0$ for $j = 2,\ldots,i$ (from Assumption 1), we have (for $j = 2,\ldots,i$)

$$\begin{aligned}&\frac{\gamma^{n-i}(\tau)}{L^{i-1}}\Big(\frac{L^{j-1}|\varepsilon_j|}{\gamma^{n-j+1}(\tau)} + \frac{L^{m_{i,j}\times(j-1)}|\varepsilon_j|^{m_{i,j}}}{\gamma^{m_{i,j}\times(n-j+1)}(\tau)}\Big) \\ &\le \frac{|\varepsilon_j|}{L^{i-j}} + \frac{|\varepsilon_j|^{m_{i,j}}}{L^{i-1-m_{i,j}\times(j-1)}} \le \frac{|\varepsilon_j|+|\varepsilon_j|^{m_{i,j}}}{L^{i-j}} \ \le\ \frac{2|\varepsilon_j|+1}{L^{i-j}}.\end{aligned}$$

Recall from (14) the special $\bar{x}_2 = L\zeta_2 - M\bar{y}\varphi^2(\cdot)$ and $\bar{x}_i = L^{i-1}\zeta_i$, $i\ge 3$. Putting them into the last term of (S.9), we can also obtain after an argument similar to the above

$$\begin{cases}\frac{\gamma^{n-i}(\tau)}{L^{i-1}}\bigg(\frac{|\bar{x}_2|}{\gamma^{n-1}(\tau)} + \frac{|\bar{x}_2|^{m_{i,2}}}{\gamma^{m_{i,2}\times(n-1)}(\tau)}\bigg) \le \frac{2|\zeta_2|+1}{L^{i-2}} + \frac{2M|\bar{y}|\varphi^2(\bar{y})+1}{L^{i-1}}, \\ \frac{\gamma^{n-i}(\tau)}{L^{i-1}}\bigg(\frac{|\bar{x}_j|}{\gamma^{n-j+1}} + \frac{|\bar{x}_j|^{m_{i,j}}}{\gamma^{m_{i,j}\times(n-j+1)}}\bigg) \le \frac{2|\zeta_j|+1}{L^{i-j}},\ \ j = 3,\ldots,i.\end{cases}$$

Feeding the estimates above back into (S.9) and using $L\ge 1$, we have

$$\begin{aligned}\|\Lambda(\cdot)f(\cdot)\| &\le \theta\bar{\phi}(\bar{y})\sum_{i=2}^{n}\bigg(\frac{2|\bar{y}|+1}{L^{i-1}} + \sum_{j=2}^{i}\Big(\frac{2|\varepsilon_j|+2|\zeta_j|+2}{L^{i-j}} \\ &\qquad + \frac{2M|\bar{y}|\varphi^2(\bar{y})+1}{L^{i-1}}\Big)\bigg) \\ &\le \theta(n-1)\bar{\phi}(\bar{y})\Big(\frac{2|\bar{y}|+1}{L} + \frac{n(2M|\bar{y}|\varphi^2(\bar{y})+1)}{2L} \\ &\qquad + n + 2\sqrt{n-1}(\|\varepsilon\|+\|\zeta\|)\Big).\end{aligned}$$

Then, noting the expression of $\beta_1(\cdot)$, we see (S.3) holds for some $\bar{\theta} > 0$. □

II. Detailed Proof of Lemma 2

Lemma 2: If high gain $L(\tau)$ or $M(\tau)$ is bounded on the maximal interval $[0,\,\tau_f)$, then the closed-loop system states $\big(\varepsilon(\tau),\,\bar{y}(\tau),\,\zeta(\tau)\big)$ and the control input $u(\tau)$ are all bounded on $[0,\,\tau_f)$.

Proof. The premise that at least one of the two high gains is bounded in effect indicates that both of them are bounded, according to Lemma 1.

Under the condition of $L(\tau)$ and $M(\tau)$ being bounded, the exposure of the boundedness of $\big(\varepsilon(\tau),\,\bar{y}(\tau),\,\zeta(\tau)\big)$ and $u(\tau)$ can be converted into that of the following auxiliary analysis states on $[0,\,\tau_f)$:

$$\begin{cases}\bar{\zeta} \ =\ \frac{\zeta}{L^*}, \\ \epsilon_1 \ =\ \frac{\bar{y}}{LL^*},\quad \epsilon_i \ =\ \frac{\varepsilon_i}{L(L^*)^i},\quad i = 2,\ldots,n,\end{cases} \quad \text{(S.10)}$$

where $L^* > 0$ is a sufficiently large analysis parameter.

For easy use, we let $L^\infty = \lim_{\tau\to\tau_f} L(\tau)$ and $M^\infty = \lim_{\tau\to\tau_f} M(\tau)$, since L and M are assumed bounded.

Consider the new composite Lyapunov function

$$V_3 = L\bar{\zeta}^{\mathrm{T}}Q\bar{\zeta} + \nu_2(L^\infty)^2LL^*\epsilon^{\mathrm{T}}S\epsilon =: V_{\bar{\zeta}} + \nu_2(L^\infty)^2V_\epsilon, \quad \text{(S.11)}$$

with $\nu_2 = 2(\|Qa\|^2+1)$ and matrix S as in (18).

Invoke the estimate results of $\frac{\mathrm{d}V_{\bar{\zeta}}}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}V_\epsilon}{\mathrm{d}\tau}$ from Props. S1 and S2 in Section III. By the assumed boundedness of L and M, it is then clear to see that

$$\begin{aligned}\frac{\mathrm{d}V_3}{\mathrm{d}\tau} \ \le\ & -L^2\|\bar{\zeta}\|^2 - \nu_2L^{\infty2}L^2L^{*2}\big(\tfrac{1}{2} - \tfrac{\theta_\epsilon}{L^*}\big)\|\epsilon\|^2 \\ & -\big(1 - \tfrac{(L^\infty)^2}{L^*}\big)\bar{\phi}^2(\bar{y})\|\bar{\zeta}\|^2 + \theta_4\zeta_2^2 + \theta_4\bar{y}^2\varphi^2 + \theta_4,\end{aligned}$$

for some unknown positive θ_4 that depends on L^∞ and M^∞.

By picking analysis parameter $L^* \ge \max\{4\theta_\epsilon,\,(L^\infty)^2\}$, the 2nd and 3rd terms above are readily rendered negative definite. Then noting from (21) that $\frac{\mathrm{d}L}{\mathrm{d}\tau} \ge \zeta_2^2 - \delta_3$ and $\frac{\mathrm{d}M}{\mathrm{d}\tau} \ge \bar{y}^2\varphi^2(\bar{y}) - \delta_4$, we are led to

$$\frac{\mathrm{d}V_3}{\mathrm{d}\tau} \le -C_4V_3 + \theta_4\frac{\mathrm{d}L}{\mathrm{d}\tau} + \theta_4\frac{\mathrm{d}M}{\mathrm{d}\tau} + \theta_4(\delta_3+\delta_4), \quad \text{(S.12)}$$

for some positive constant C_4.

Solving differential inequality (S.12), as similarly done for (25), gives the boundedness of $V_3(\tau)$ at once. It is then obvious that $\bar{\zeta}(\tau)$ and $\epsilon(\tau)$ are both bounded on $[0,\,\tau_f)$.

By the definitions of $\bar{\zeta}$ and ϵ in (S.10), we directly arrive at the boundedness of $\zeta_i(\tau)$'s, $\bar{y}(\tau)$ and $\varepsilon_i(\tau)$'s.

The proof is thus completed. □

III. Properties of Auxiliary Analysis States in (S.10)

We recall the auxiliary analysis states from (S.10) and present two important propositions to serve the analysis in Section VI. The true value of the analysis states lies in the case when the dynamic-high-gain ISpS property of $(\varepsilon,\,\zeta)$ ceases to hold due to the boundedness of $L(\tau)$ preventing $L(\tau)$ itself from growing to arbitrarily large. By appealing to an analysis parameter L^* to rescale $(\zeta,\,\bar{y},\,\varepsilon)$, the dynamic-high-gain ISpS is converted into the conventional ISpS which largely serves the establishment of system boundedness (see, i.e., Lemma 2).

Note that the rescaling (i.e., (S.10)) follows the similar idea in [17]–[19]. But the essential difference consists in which variables to be rescaled and how to rescale them with suitable factors/parameters. Moreover, deriving their important ISpS property certifies the selection of design functions $\beta_i(\cdot)$'s in (16).

For $\bar{\zeta}$ in (S.10), we have (with the aid of $\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}$ (21))

$$\begin{aligned}\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\tau} \ =\ & LK\bar{\zeta} + L^2L^*a\epsilon_2 + \tfrac{a}{L^*}f_1\big(\mu^{-1}(\tau),x\big)\gamma^{n-1}(\tau) \\ & + \bar{D}_{n-1}\bar{\zeta}\frac{\gamma'(\tau)}{\gamma(\tau)} - D_{n-1}\bar{\zeta}\frac{\mathrm{d}L}{\mathrm{d}\tau}\cdot\frac{1}{L} + \frac{\Xi(\cdot)}{L^*}. \quad \text{(S.13)}\end{aligned}$$

In comparison with ε, new scaled observer error $\epsilon = [\epsilon_1,\,\ldots,\epsilon_n]^{\mathrm{T}}$, no longer $(n-1)$-dimensional, expands its dimension to n with a scaled quantity $\epsilon_1 = \frac{\bar{y}}{LL^*}$. Differently from $\bar{\zeta}$, error ϵ is more importantly obtained via a dynamic scaling $\frac{1}{L(\tau)(L^*)^i}$. As a result, its dynamics are actually quite different from those of ε (see (9)) and need recomputing:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = LL^*A_l\,\epsilon + lLL^*\epsilon_1 - L\bar{a}\epsilon_2 + \bar{\Lambda}(\gamma(\tau), LL^*)\bar{f}\big(\mu^{-1}(\tau), x\big) - \frac{\bar{a}\gamma^{n-1}(\tau)}{(LL^*)^2}f_1(\cdot) + \big[\bar{\zeta}_2 - \frac{M\bar{y}\varphi^2(\bar{y})}{LL^*},\ \mathbf{0}_{1\times(n-1)}\big]^{\mathrm{T}} + \bar{D}_n\epsilon\frac{\gamma'(\tau)}{\gamma(\tau)} - D_n\epsilon\frac{\mathrm{d}L}{\mathrm{d}\tau}\cdot\frac{1}{L}, \tag{S.14}$$

where $A_l = \big[-l,\, [\mathbf{I}_{n-1}, \mathbf{0}_{(n-1)\times 1}]^{\mathrm{T}}\big]$ with parameter vector $l = [l_1, \dots, l_n]^{\mathrm{T}}$, $\bar{a} = \big[0, a_2, \frac{a_3}{L^*}, \dots, \frac{a_n}{(L^*)^{n-2}}\big]^{\mathrm{T}}$, $\bar{\Lambda}(\gamma(\tau), LL^*) = \mathbf{diag}\big\{\frac{\gamma^{n-1}(\tau)}{LL^*}, \frac{\gamma^{n-2}(\tau)}{(LL^*)^2}, \dots, \frac{1}{(LL^*)^n}\big\}$, $\bar{f}(\cdot) = \big[f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)\big]^{\mathrm{T}}$, $\bar{D}_n = \mathbf{diag}\{n, n-1, \dots, 1\}$ and $D_n = \mathbf{diag}\{1, 2, \dots, n\}$.

Note that in (S.14), vector l is chosen such that A_l is Hurwitz and meanwhile such that there is

$$\begin{cases} A_l^{\mathrm{T}}S + SA_l \le -2\mathbf{I}_n, \\ \underline{c}_3\mathbf{I}_n \le S\big(D_n - \frac{1}{2}\mathbf{I}_n\big) + \big(D_n - \frac{1}{2}\mathbf{I}_n\big)^{\mathrm{T}}S \le \bar{c}_3\mathbf{I}_n, \end{cases} \tag{18}$$

for a symmetric positive definite matrix S, and positive constants $\underline{c}_3$ and $\bar{c}_3$ satisfying $\underline{c}_3 < \bar{c}_3$.

Now with (S.13) and (S.14) in hand, we consider Lyapunov function candidates $V_{\bar{\zeta}} = L\bar{\zeta}^{\mathrm{T}}Q\bar{\zeta}$ and $V_\epsilon = LL^*\epsilon^{\mathrm{T}}S\epsilon$. Their derivative estimate results are presented in the following propositions which suggest certain ISpS properties associated with $\bar{\zeta}$ and ϵ.

Proposition S1: Let $V_{\bar{\zeta}} = L\bar{\zeta}^{\mathrm{T}}Q\bar{\zeta}$ with Q satisfying (15). Then along $\bar{\zeta}$-system (S.13), there is

$$\frac{\mathrm{d}V_{\bar{\zeta}}}{\mathrm{d}\tau} \le -L^2\|\bar{\zeta}\|^2 + \frac{\nu_2}{2}L^4L^{*2}\|\epsilon\|^2 - \bar{\phi}^2\|\bar{\zeta}\|^2 + \big(\tfrac{L}{L^*}\big)^2\zeta_2^2 + \frac{n^2+1+\|Q\|^2}{L^{*2}}\bar{y}^2\varphi^2(\bar{y}) + \frac{(\theta\|Qa\|L)^2 + (\theta\|Q\|M)^2}{L^{*2}}, \tag{S.15}$$

with $\nu_2 = 2(2\|Qa\|^2 + 1)$ as before. This suggests analysis state $\bar{\zeta}$ is *ISpS* with respect to $\big(\|\epsilon\|^2, \zeta_2^2, \bar{y}^2\varphi^2(\bar{y})\big)$, provided that high gain $L(\tau)$ is bounded.

Proof. See the end of this section for details.

Proposition S2: Let $V_\epsilon = LL^*\epsilon^{\mathrm{T}}S\epsilon$ with S satisfying (18). Then along ϵ-system (S.14), there is for some unknown θ_ϵ

$$\frac{\mathrm{d}V_\epsilon}{\mathrm{d}\tau} \le -L^2L^{*2}\big(1 - \tfrac{\theta_\epsilon}{L^*}\big)\|\epsilon\|^2 + \frac{\bar{\phi}^2(\bar{y})}{\nu_2 L^*}\|\bar{\zeta}\|^2 + \zeta_2^2 + \theta_\epsilon\bar{y}^2\varphi^2(\bar{y}) + \theta_\epsilon, \tag{S.16}$$

which indicates ϵ is *ISpS* regarding $\big(\bar{\phi}^2(\bar{y})\|\bar{\zeta}\|^2, \zeta_2^2, \bar{y}^2\varphi^2(\bar{y})\big)$, provided that $L^* > 2\theta_\epsilon$.

Proof. Note that the proposition can be shown in an analogous fashion to the proof of Prop. S1. Thus, we omit its proof here while we provide the detailed proof of Prop. S1 below. It is noted that to obtain (S.16), $\bar{D}_nS + S\bar{D}_n \le c_3\mathbf{I}_n$ for some $c_3 > 0$ (from (19)) is used and that $-\frac{\mathrm{d}L}{\mathrm{d}\tau} \le \delta_1L^2 - \delta_2L\beta_5(M, \bar{y})$ with $\beta_5(\cdot)$ is tailored to eliminating the nonlinearities. ∎

Proof of Prop. S1. Taking time derivative of $V_{\bar{\zeta}}$ along the solutions of $\bar{\zeta}$-system (S.13), we have

$$\frac{\mathrm{d}V_{\bar{\zeta}}}{\mathrm{d}\tau} = L^2\bar{\zeta}^{\mathrm{T}}(K^{\mathrm{T}}Q + QK)\bar{\zeta} + 2L^3L^*\epsilon_2\bar{\zeta}^{\mathrm{T}}Qa + \frac{2L}{L^*}f_1(\cdot)\gamma^{n-1}(\tau)\bar{\zeta}^{\mathrm{T}}Qa + L\bar{\zeta}^{\mathrm{T}}(\bar{D}_{n-1}^{\mathrm{T}}Q + Q\bar{D}_{n-1})\bar{\zeta}\frac{\gamma'(\tau)}{\gamma(\tau)} + 2L\bar{\zeta}Q\frac{\Xi(\cdot)}{L^*} - \bar{\zeta}^{\mathrm{T}}\big((D_{n-1} - \tfrac{\mathbf{I}_{n-1}}{2})^{\mathrm{T}}Q + Q(D_{n-1} - \tfrac{\mathbf{I}_{n-1}}{2})\big)\bar{\zeta}\frac{\mathrm{d}L}{\mathrm{d}\tau}. \tag{S.17}$$

Invoke (15) and note $-\frac{\mathrm{d}L}{\mathrm{d}\tau} \le \delta_1L^2 - \delta_2L\big(\beta_3(\cdot) + \beta_4(\cdot)\big)$ in (21). Then using $\delta_1\bar{c}_2 \le \frac{1}{4}$ and $\delta_2\underline{c}_2 \ge 1$, we see the first and last terms in (S.17) satisfy

$$\begin{cases} L^2\bar{\zeta}^{\mathrm{T}}(K^{\mathrm{T}}Q + QK)\bar{\zeta} \le -2L^2\|\bar{\zeta}\|^2, \\ -\bar{\zeta}^{\mathrm{T}}\big((D_{n-1} - \frac{\mathbf{I}_{n-1}}{2})^{\mathrm{T}}Q + Q(D_{n-1} - \frac{\mathbf{I}_{n-1}}{2})\big)\bar{\zeta}\frac{\mathrm{d}L}{\mathrm{d}\tau} \\ \qquad \le \frac{L^2}{4}\|\bar{\zeta}\|^2 - L\big(\beta_3(\cdot) + \beta_4(\cdot)\big)\|\bar{\zeta}\|^2. \end{cases}$$

Putting the above estimates into (S.17) yields

$$\frac{\mathrm{d}V_{\bar{\zeta}}}{\mathrm{d}\tau} \le -\frac{7}{4}L^2\|\bar{\zeta}\|^2 - L\big(\beta_3(\cdot) + \beta_4(\cdot)\big)\|\bar{\zeta}\|^2 + \frac{2L}{L^*}f_1(\cdot)\gamma^{n-1}(\tau)\bar{\zeta}^{\mathrm{T}}Qa + 2L^3L^*\epsilon_2\bar{\zeta}^{\mathrm{T}}Qa + L\bar{\zeta}^{\mathrm{T}}(\bar{D}_{n-1}^{\mathrm{T}}Q + Q\bar{D}_{n-1})\bar{\zeta}\frac{\gamma'(\tau)}{\gamma(\tau)} + 2L\bar{\zeta}Q\frac{\Xi(\cdot)}{L^*}. \tag{S.18}$$

Recall from (S.6) that $f_1(\cdot)\gamma^{n-1}(\tau) \le \theta\bar{\phi}(\bar{y})(2|\bar{y}|+1)$. Then by completing the square, we see the 3rd and 4th terms of (S.18) satisfy

$$\begin{cases} \frac{2L}{L^*}f_1\gamma^{n-1}(\tau)\bar{\zeta}^{\mathrm{T}}Qa \le \bar{\phi}^2(\bar{y})(2|\bar{y}|+1)^2\|\bar{\zeta}\|^2 + \big(\frac{\theta\|Qa\|L}{L^*}\big)^2, \\ 2L^3L^*\epsilon_2\bar{\zeta}^{\mathrm{T}}Qa \le \frac{L^2}{2}\|\bar{\zeta}\|^2 + 2\|Qa\|^2L^4L^{*2}\|\epsilon\|^2. \end{cases}$$

Noting from (19) that $\bar{D}_{n-1}Q + Q\bar{D}_{n-1} \le c_2\mathbf{I}_{n-1}$ for some $c_2 > 0$, we find (by using $\frac{\gamma'(\tau)}{\gamma(\tau)} \le 1$ in (4) and $L(\tau) \ge L(0) \ge 4c_2$)

$$L\bar{\zeta}^{\mathrm{T}}(\bar{D}_{n-1}^{\mathrm{T}}Q + Q\bar{D}_{n-1})\bar{\zeta}\frac{\gamma'(\tau)}{\gamma(\tau)} \le c_2L\|\bar{\zeta}\|^2 \le \frac{L^2}{4}\|\bar{\zeta}\|^2.$$

As for the last term in (S.18), we recall the $\Xi(\cdot)$ defined in (20) and transform it into the τ-horizon by use of $\frac{\dot{M}}{\mu(t)} = \frac{\mathrm{d}M}{\mathrm{d}\tau}$, $\frac{\dot{\bar{y}}}{\mu(t)} = \frac{\mathrm{d}\bar{y}}{\mathrm{d}\tau}$, $\frac{\dot{\mu}(t)}{\mu(t)^2} = \frac{\gamma'(\tau)}{\gamma(\tau)} \le 1$. It is then not difficult to obtain (by invoking $L\varepsilon_2 = (LL^*)^2\epsilon_2$ and $\bar{x}_2 = L\zeta_2 - M\bar{y}\varphi^2(\bar{y})$)

$$\begin{aligned} 2L\bar{\zeta}Q\frac{\Xi}{L^*} \le\ & 2\|\bar{\zeta}\|\frac{\|Q\|}{L^*}|\bar{y}|\varphi^2(\bar{y})\big(\tfrac{\mathrm{d}M}{\mathrm{d}\tau} + 2M\big) \\ & + 2\|\bar{\zeta}\|\frac{\|Q\|M}{L^*}\Big|\frac{\partial(\bar{y}\varphi^2)}{\partial\bar{y}}\Big|\cdot|(LL^*)^2\epsilon_2 + L\zeta_2| \\ & + 2\|\bar{\zeta}\|\frac{\|Q\|M}{L^*}\Big|\frac{\partial(\bar{y}\varphi^2)}{\partial\bar{y}}\Big|(n|\bar{y}| + M|\bar{y}|\varphi^2 + |f_1|\gamma^{n-1}) \\ =:\ & \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Further estimating its three terms by use of the same reasoning as above, we complete the square and learn (noting $\varphi(\cdot) \ge 1$)

$$\begin{cases} \mathrm{I} \le \varphi^2(\bar{y})\big(\frac{\mathrm{d}M}{\mathrm{d}\tau} + 2M\big)^2\|\bar{\zeta}\|^2 + \big(\frac{\|Q\|^2}{L^*}\big)^2\bar{y}^2\varphi^2(\bar{y}), \\ \mathrm{II} \le L^4L^{*2}\|\epsilon\|^2 + 2\|Q\|^2M^2\big(\frac{\partial(\bar{y}\varphi^2)}{\partial\bar{y}}\big)^2\|\bar{\zeta}\|^2 + \big(\frac{L}{L^*}\big)^2\zeta_2^2, \\ \mathrm{III} \le \|Q\|^2M^2(1 + M^2)\big(\frac{\partial(\bar{y}\varphi^2)}{\partial\bar{y}}\big)^2\|\bar{\zeta}\|^2 + \frac{n^2+1}{(L^*)^2}\bar{y}^2\varphi^2 \\ \qquad + \bar{\phi}^2(2|\bar{y}| + 1)^2\big(\frac{\partial(\bar{y}\varphi^2)}{\partial\bar{y}}\big)^2\|\bar{\zeta}\|^2 + \big(\frac{\theta\|Q\|M}{L^*}\big)^2. \end{cases}$$

Now putting the above estimates into (S.17) and noting the expressions of $\beta_3(\cdot)$ and $\beta_4(\cdot)$ in (16) immediately lead to (S.15). ∎

IV. Detailed Proofs of Lemma 4 and Theorem 3 about Uniform Boundedness

Before presenting the proofs of Lemma 4 and Theorem 3, we propose to give two crucial technical propositions which, aiming at a concrete solution, provide two delicate estimates of V_1 on different intervals of interest. Based on the propositions, we then conduct, via exhaustive Lyapunov analysis, the detailed proofs of Lemma 4 and Theorem 3.

Proposition S3: For the solution $X = (\varepsilon, \bar{y}, \bar{x}, L, M)$ in τ-horizon starting from the given initial value $X(0) = (\varepsilon(0), \bar{y}(0), \bar{x}(0), L(0), M(0))$, there is the following estimate on any of its finite existence interval $[0, \tau_e]$,

$$V_1(\tau) \le E_1\big(X(0), L(\tau_e)\big), \tag{S.19}$$

where $V_1 = \nu_1V_\varepsilon + V_\zeta + \frac{1}{2}\bar{y}^2$ as before, and $E_1(\cdot)$ is a nonnegative continuous function.

Proof. We postpone the detailed proof to the end of this section for a clear line of thought. ∎

Proposition S4: If $\frac{\mathrm{d}V_1}{\mathrm{d}\tau} \le -C_1 V_1(\tau) + \theta_1 \frac{\mathrm{d}M}{\mathrm{d}\tau} + \theta_2$ (i.e., (25)) holds on some time interval $[\underline{\tau},\, \overline{\tau}]$, then there is

$$V_1(\tau) \;\le\; E_2\big(X(0),\, L(\underline{\tau}),\, M(\overline{\tau})\big), \quad \forall \tau \in [0,\, \overline{\tau}], \quad \text{(S.20)}$$

for some nonnegative continuous function $E_2(\cdot)$. Particularly, if (25) holds on $[\underline{\tau},\, +\infty)$, then $(S.20)$ with $M(\overline{\tau}) = M^{\infty}$ holds on $[0,\, +\infty)$.

Proof. On $[\underline{\tau},\, \overline{\tau}]$, solving (25) yields

$$V_1(\tau) \le V_1(\underline{\tau}) + \max\{\theta_1,\, \theta_2\}\big(M(\overline{\tau}) - M(\underline{\tau}) + \tfrac{1}{C_1}\big). \quad \text{(S.21)}$$

By Prop. S3, there is $V_1(\tau) \;\le\; E_1\big(X(0),\, L(\underline{\tau})\big)$, $\forall \tau \in [0,\, \underline{\tau}]$. This together with (S.21) immediately gives

$$\begin{aligned} V_1(\tau) \;&\le\; E_1\big(X(0),\, L(\underline{\tau})\big) + \max\{\theta_1,\, \theta_2\}\big(M(\overline{\tau}) + \tfrac{1}{C_1}\big) \\ &=:\; E_2\big(X(0),\, L(\underline{\tau}),\, M(\overline{\tau})\big). \end{aligned}$$

The result on $[\underline{\tau},\, +\infty)$ holds obviously by following the analogous reasoning. □

Lemma 4: Given an initial value, the possible infinitely many solutions $L(\tau)$ are uniformly bounded on $[0,\, +\infty)$, if and only if the possible infinitely many $M(\tau)$ (starting from the same initial value) are also uniformly bounded on $[0,\, +\infty)$.

Proof. The proof is lengthy and hence deferred to the end of this section. □

Lemma 5: Given an initial value, if the possible infinitely many solutions $L(\tau)$ or $M(\tau)$ are uniformly bounded on $[0,\, +\infty)$, then the closed-loop states $\big(\varepsilon(\tau),\, \bar{y}(\tau),\, \zeta(\tau)\big)$ and the control input $u(\tau)$ are all uniformly bounded.

Proof. The lemma can be proved in an analogous manner to that of Lemma 2 and hence we omit its proof. □

Theorem 3: Consider the resulting closed-loop system consisting of (1), (5), (13), (14) and (16) in finite t-horizon. The possible infinitely many solutions starting from the same given initial value are uniformly bounded on the arbitrarily prescribed $[0,\, T)$.

Proof. It suffices, in light of the arguments in Section VI of the paper, to confine ourselves to the closed-loop system (21) in infinite τ-horizon and disclose the uniform boundedness of its possible infinitely many solutions on $[0,\, +\infty)$.

Keeping in mind Lemmas 4 and 5, we only need to show the uniform boundedness of possible infinitely many solutions $M(\tau)$. In doing so, we take $V_1 = \nu_1 V_\varepsilon + V_\zeta + \frac{1}{2}\bar{y}^2$ once again, estimate its time derivative as before to obtain (24) and take (24) as a starting point.

We suppose, given an initial value, the possible infinitely many solutions $M(\tau)$ starting from it *do not share a uniform bound* (though each one of them is bounded due to Theorem 1). Then solutions $L(\tau)$ are not uniformly bounded either according to its dynamics in (21). Naturally, among all the possible infinitely many solutions $M(\tau)$, there always exists one with a large bound satisfying

$$M^{\infty} \;>\; \max\big\{\sqrt{16\theta_\varepsilon},\, 2\theta_1,\, \tfrac{2\theta_1}{\lambda C_3}\big\} \;=:\; \bar{\theta}_1.$$

Then, if $M(0) < \bar{\theta}_1$, an instant $\tau_1 > 0$ exists such that $M(\tau_1) = \bar{\theta}_1$ while for $M(0) \ge \bar{\theta}_1$, we let $\tau_1 = 0$. In view of this, we have $M(\tau) \ge \bar{\theta}_1$, $\forall \tau \ge \tau_1$.

In light of $\varphi(\bar{y}) \ge 1$ and $\frac{\mathrm{d}L}{\mathrm{d}\tau} \ge \frac{\mathrm{d}M}{\mathrm{d}\tau}$ with $L(0) \ge M(0)$, it is straightforward to infer from (24) that, like (28),

$$\begin{aligned} \tfrac{\mathrm{d}V_1}{\mathrm{d}\tau} \;&\le\; -C_3 M V_1(\tau) + \theta_1 \;\le\; -\tfrac{2\theta_1}{\lambda} V_1(\tau) + \theta_1, \\ &\qquad \forall \tau \in [\tau_1,\, +\infty). \end{aligned} \quad \text{(S.22)}$$

Though (S.22) is in an analogous form to (25), direct use of Prop. S4 would lead to $E_2(\cdot)$ depending on $L(\tau_1)$ and M^{∞}. This is unwanted, because the size of $L(\tau_1)$ and M^{∞} would differ from solutions to solutions. To get over the dependence, we solve (S.22)

$$\begin{aligned} V_1(\tau) \;&\le\; e^{-\frac{2\theta_1}{\lambda}(\tau - \tau_1)} V_1(\tau_1) + \tfrac{\lambda}{2}\big(1 - e^{-\frac{2\theta_1}{\lambda}(\tau - \tau_1)}\big), \\ &\qquad \forall \tau \in [\tau_1,\, +\infty). \end{aligned} \quad \text{(S.23)}$$

Similarly, the first term can reduce to less than $\frac{\lambda}{2}$ after a finite time duration:

$$\bar{\Delta} \;:=\; \max\big\{0,\, \tfrac{\lambda}{2\theta_1}\ln\tfrac{2V_1(\tau_1)}{\lambda}\big\}.$$

To get rid of the influence of τ_1 in $\bar{\Delta}$, we need to give a suitable estimate of $V_1(\tau_1)$. Note that $M(\tau_1) = \bar{\theta}_1$. According to the size of $L(0)$, the discussions are split into two scenarios: $L(0) \ge \bar{\theta}_1$ and $L(0) < \bar{\theta}_1$.

(i) $\boldsymbol{L(0) \ge \bar{\theta}_1}$. In this case, (25) holds on the whole $[0,\, \tau_1]$. Employing Prop. S4 directly with $\underline{\tau} = 0$ and $\overline{\tau} = \tau_1$, we see from the definition of $\bar{\theta}_1$ that

$$V_1(\tau) \;\le\; E_2\big(X(0),\, L(0),\, \bar{\theta}_1\big), \quad \forall \tau \in [0,\, \tau_1].$$

(ii) $\boldsymbol{L(0) < \bar{\theta}_1}$. In this case, by $L(\tau_1) \ge M(\tau_1) = \bar{\theta}_1$, there always exists a time instant τ_2 ($0 < \tau_2 \le \tau_1$) such that $L(\tau_2) = \bar{\theta}_1$; thus, (25) holds on $[\tau_2,\, \tau_1]$. By applying Prop. S4 with $\underline{\tau} = \tau_2$ and $\overline{\tau} = \tau_1$, it is clear that

$$V_1(\tau) \;\le\; E_2\big(X(0),\, \bar{\theta}_1,\, \bar{\theta}_1\big), \quad \forall \tau \in [0,\, \tau_1].$$

Synthesizing both cases, we see $V_1(\tau_1) \le \bar{E}_2\big(X(0),\, \bar{\theta}_1\big)$, for a nonnegative continuous function $\bar{E}_2(\cdot)$ independent of the solution itself. This leads to

$$\bar{\Delta} \;\le\; \max\big\{0,\, \tfrac{\lambda}{2\theta_1}\ln\tfrac{2\bar{E}_2\left(X(0),\, \bar{\theta}_1\right)}{\lambda}\big\} =: \tilde{\Delta}.$$

Thus, after $M(\tau)$ reaches $\bar{\theta}_1$, it takes $V_1(\tau)$ at most a "$\tilde{\Delta}$" amount of time to reduce to less than λ.

By $V_1 = \nu_1 V_\varepsilon + V_\zeta + \frac{1}{2}\bar{y}^2$, it is obvious to learn $\bar{y}^2(\tau) < 2\lambda$, $\forall \tau \ge \tau_1 + \tilde{\Delta}$. Then from $\frac{\mathrm{d}M}{\mathrm{d}\tau} = \max\big\{\bar{y}^2\varphi^2(\bar{y}) - \delta_4,\, 0\big\}$ and the increasing property of $\varphi(\cdot)$ regarding $|\bar{y}|$, one can always, by selecting λ sufficiently small, make sure

$$\bar{y}^2(\tau)\varphi^2(\bar{y}) - \delta_4 \le 2\lambda\varphi^2\big(\sqrt{2\lambda}\big) - \delta_4 \le 0, \;\; \forall \tau \ge \tau_1 + \tilde{\Delta},$$

which leads to

$$\tfrac{\mathrm{d}M}{\mathrm{d}\tau} \;\equiv\; 0, \quad \forall \tau \ge \tau_1 + \tilde{\Delta}.$$

On interval $[\tau_1,\, \tau_1 + \tilde{\Delta}]$, observe from (S.23) and $V_1(\tau_1) \le \bar{E}_2\big(X(0),\, \bar{\theta}_1\big)$ that

$$\bar{y}^2(\tau) \;\le\; 2V_1(\tau) \;\le\; 2\big(\bar{E}_2\big(X(0),\, \bar{\theta}_1\big) + \tfrac{\lambda}{2}\big) \;=:\; \tilde{\theta}_1.$$

Then, the incremental of $M(\tau)$ in the duration $\tilde{\Delta}$ can be estimated as follows

$$M(\tau_1 + \tilde{\Delta}) - M(\tau_1) \;=\; \int_{\tau_1}^{\tau_1 + \tilde{\Delta}} \mathrm{d}M \;\le\; \tilde{\Delta}\tilde{\theta}_1\varphi^2(\sqrt{\tilde{\theta}_1}).$$

At this stage, we have obtained

$$M(\tau) \;\le\; \begin{cases} \bar{\theta}_1, & 0 \le \tau \le \tau_1, \\ \bar{\theta}_1 + \tilde{\Delta}\tilde{\theta}_1\varphi^2(\sqrt{\tilde{\theta}_1}), & \tau \ge \tau_1, \end{cases}$$

which shows that the bound of $M(\tau)$ depends merely on initial data and system unknown parameters.

We thus conclude that all the possible infinitely many solutions $M(\tau)$ starting from the same given initial value are uniformly bounded, and so are the closed-loop system by use of Lemmas 4 and 5. □

Proof of Prop. S3. Let us work with a new $\bar{V}_3 = V_{\bar{\zeta}} + \nu_2 L^2(\tau_e) V_\epsilon$, which parallels the V_3 in (S.11), except for the weight being replaced with $\nu_2 L^2(\tau_e)$.

Analogously, invoke the estimate results of $\frac{\mathrm{d}V_{\bar{\zeta}}}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}V_\epsilon}{\mathrm{d}\tau}$ from Props. S1 and S2 in Section III. Noting the boundedness of L and M on $[0,\,\tau_e]$ and $\frac{\mathrm{d}L}{\mathrm{d}\tau} \geq \frac{\mathrm{d}M}{\mathrm{d}\tau}$, we can then obtain by picking a new analysis parameter $L^* \geq \max\{4\theta_\epsilon,\, L^2(\tau_e)\}$

$$\frac{\mathrm{d}\bar{V}_3}{\mathrm{d}\tau} \;\leq\; -C_4 \bar{V}_3(\tau) + 2\theta_4^* \frac{\mathrm{d}L}{\mathrm{d}\tau} + 2\theta_4^* \delta_3, \quad \forall \tau \in [0,\,\tau_e], \quad \text{(S.24)}$$

with C_4 as in (S.12) but θ_4^* no longer depending on L^∞ but depending on $L(\tau_e)$ instead.

Solving (S.24) directly yields

$$\begin{aligned}\bar{V}_3(\tau) \;&\leq\; \bar{V}_3(0) + 2\theta_4^* \max\{1, \delta_3\}\big(L(\tau_e) + \tfrac{1}{C_4}\big)\\ &=:\; \bar{E}_1(\bar{V}_3(0),\, L(\tau_e)), \quad \forall \tau \in [0,\,\tau_e].\end{aligned}$$

It is then natural from the facts that $V_\zeta = L^{*2} V_{\bar{\zeta}} \leq L^{*2} \bar{V}_3$ and $\lambda_{\min}(P)\big(\frac{\bar{y}^2}{LL^*} + \frac{\|\varepsilon\|^2}{LL^{*2n-1}}\big) \leq LL^* \epsilon^{\mathrm{T}} P \epsilon \leq \frac{\bar{V}_3}{\nu_2 L^2(\tau_e)}$ to see

$$\begin{cases} V_\zeta(\tau) \;\leq\; L^{*2} \bar{E}_1(\bar{V}_3(0),\, L(\tau_e)),\\ V_\varepsilon(\tau) \leq \lambda_{\max}(P) L \|\varepsilon\|^2 \leq \frac{\lambda_{\max}(P) L^{*2n-1}}{\nu_2 \lambda_{\min}(P)} \bar{E}_1(\bar{V}_3(0), L(\tau_e)),\\ \frac{1}{2}\bar{y}^2(\tau) \;\leq\; \frac{L^*}{\nu_2 \lambda_{\min}(P) L(\tau_e)} \bar{E}_1(\bar{V}_3(0),\, L(\tau_e)). \end{cases}$$

Thus, recalling $V_1 = \nu_1 V_\varepsilon + V_\zeta + \frac{1}{2}\bar{y}^2$, we arrive at (S.19) at once. □

Proof of Lemma 4. Necessity part, i.e., a uniformly bounded $L(\tau)$ leads to a uniformly bounded $M(\tau)$, holds trivially since there always exists $L(\tau) \geq M(\tau)$ according to the facts that $\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} \geq \frac{\mathrm{d}M(\tau)}{\mathrm{d}\tau}$ and $L(0) \geq M(0)$.

In contrast, sufficiency part, though in a similar flavor to Lemma 1, is proved based on a different angle in contradictory arguments — the sufficient largeness of the bound of $L(\tau)$.

Let us work with $V_1 = \nu_1 V_\varepsilon + V_\zeta + \frac{1}{2}\bar{y}^2$ once again and estimate its time derivative as before to obtain (24). Aiming at (24), we begin to show the sufficiency — a uniformly bounded $M(\tau)$ implies a uniformly bounded $L(\tau)$ — by using a contradictory argument.

On the condition that $M(\tau)$ is uniformly bounded on $[0,\,+\infty)$ (denote by $\bar{M}$ the uniform bound), we assume the possible infinitely many solutions $L(\tau)$ which start from the same given initial value *do not share* a uniform bound, though each of them has its own bound on $[0, +\infty)$ according to Theorem 1.

This implies that there always exists a bounded $L(\tau)$ satisfying

$$L^\infty \;>\; \max\big\{\sqrt{16\theta_\varepsilon},\, \tfrac{2\theta_3}{\lambda C_2}\big\} \;=:\; \bar{\theta}_3.$$

It is then clear that if $L(0) < \bar{\theta}_3$, a finite time τ_1 must exist such that $L(\tau_1) = \bar{\theta}_3$, otherwise, we let $\tau_1 = 0$. In both cases, $\frac{\mathrm{d}V_1}{\mathrm{d}\tau}$ in (24) can be reduced to

$$\frac{\mathrm{d}V_1}{\mathrm{d}\tau} \;\leq\; -C_1 V_1(\tau) + \theta_1 \frac{\mathrm{d}M}{\mathrm{d}\tau} + \theta_1, \quad \forall \tau \in [\tau_1,\, +\infty).$$

Now we employ Prop. S4 with $\underline{\tau} = \tau_1$ and $\overline{\tau} = +\infty$ and use $L(\tau_1) = \bar{\theta}_3$ which is a constant depending merely on unknown system parameters. It is then natural to see a continuous nonnegative function $E_2(\cdot)$ exists such that

$$V_1(\tau) \;\leq\; E_2\big(X(0),\, \bar{\theta}_3,\, \bar{M}\big), \quad \forall \tau \in [0,\, +\infty), \quad \text{(S.25)}$$

which gives the uniform boundedness of $V_1(\tau)$ and in turn uniformly bounded $\bar{y}(\tau)$ and $\zeta_2(\tau)$, all on $[0,\, +\infty)$.

Recall the $\beta(M, \bar{y})$ in $\frac{\mathrm{d}L}{\mathrm{d}\tau}$ (see (16)). It is obvious to know the uniform boundedness of $\beta(M, \bar{y})$ on $[0,\, +\infty)$ from the both uniformly bounded $M(\tau)$ and $\bar{y}(\tau)$. For easy reference, we denote by $\bar{\beta}$ the uniform bound.

Since it is assumed at the very beginning of the sufficiency proof that the possible $L(\tau)$ do not share a uniform bound, we know there is one $L(\tau)$ with its large bound L^∞ satisfying

$$L^\infty \;>\; \max\big\{\tfrac{\delta_2 \bar{\beta}}{\delta_1},\, \bar{\theta}_3\big\} \;=:\; \tilde{\theta}_3.$$

Again, if $L(0) < \tilde{\theta}_3$, a finite time τ_2 must exist such that $L(\tau_2) = \tilde{\theta}_3$, otherwise, we let $\tau_2 = 0$. Examining the first component of $\frac{\mathrm{d}L}{\mathrm{d}\tau}$ and noting $\frac{\mathrm{d}L}{\mathrm{d}\tau} \geq 0$, we see $-\delta_1 L^2 + \delta_2 L \beta\big(M, \bar{y}\big) \leq 0$, $\forall \tau \in [\tau_2,\, +\infty)$. Thus, the dynamics of $L(\tau)$ reduce to

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} \;=\; \max\big\{\tfrac{\mathrm{d}M}{\mathrm{d}\tau},\, \zeta_2^2(\tau) - \delta_3\big\}, \quad \forall \tau \in [\tau_2,\, +\infty). \quad \text{(S.26)}$$

Now we consider $V_2 = V_\zeta + \nu_1 V_\varepsilon = V_1 - \frac{1}{2}\bar{y}^2$ as before. Then, similar to (27), it follows from $L(\tau) \geq \bar{\theta}_3 \geq \frac{2\theta_3}{\lambda C_2}$, $\forall \tau \geq \tau_2$ that

$$\frac{\mathrm{d}V_2}{\mathrm{d}\tau} \;\leq\; -C_2 L V_2 + \theta_3 \;\leq\; -\frac{2\theta_3}{\lambda} V_2(\tau) + \theta_3, \quad \forall \tau \geq \tau_2.$$

Solving the differential inequality immediately yields

$$\begin{aligned} V_2(\tau) \;&\leq\; e^{-\frac{2\theta_3}{\lambda}(\tau - \tau_2)} V_2(\tau_2) + \tfrac{\lambda}{2}\big(1 - e^{-\frac{2\theta_3}{\lambda}(\tau - \tau_2)}\big),\\ &\forall \tau \in [\tau_2,\, +\infty). \end{aligned} \quad \text{(S.27)}$$

Note that, as argued above, (S.25) holds as long as $L^\infty > \bar{\theta}_3$. Then from $V_2 = V_1 - \frac{1}{2}\bar{y}^2$, it is evident that

$$V_2(\tau) \;\leq\; E_2\big(X(0),\, \bar{\theta}_3,\, \bar{M}\big), \quad \forall \tau \in [0,\, +\infty).$$

Keeping this in mind, we learn the first term of (S.27) can be rendered less than $\frac{\lambda}{2}$ after a finite time duration Δ:

$$\Delta \;=\; \max\big\{0,\, \tfrac{\lambda}{2\theta_3} \ln \tfrac{2E_2(X(0),\, \bar{\theta}_3,\, \bar{M})}{\lambda}\big\},$$

which relies merely on the initial value, the given accuracy, the unknown system parameters and the uniform bound of $M(\tau)$. This, together with (S.27), indicates that after $L(\tau)$ grows to $\tilde{\theta}_3$, it will take $V_2(\tau)$ at most a "Δ" amount of time to reduce to less than λ.

Since $\zeta_2^2(\tau) \leq \frac{V_2(\tau)}{\lambda_{\min}(Q)} < \frac{\lambda}{\lambda_{\min}(Q)}$, $\forall \tau \geq \tau_2 + \Delta$ and λ can be made arbitrarily small, we naturally have $\zeta_2^2 \leq \delta_3$ after $\tau_2 + \Delta =: \tau_3$, which leads to

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} \;=\; \frac{\mathrm{d}M}{\mathrm{d}\tau}, \quad \tau \in [\tau_3,\, +\infty).$$

At this stage, we have obtained that (noting $\frac{\mathrm{d}L}{\mathrm{d}\tau} \geq 0$)

$$L(\tau) \;\leq\; \begin{cases} \tilde{\theta}_3, & 0 \leq \tau \leq \tau_2,\\ L(\tau_3), & \tau_2 < \tau \leq \tau_3,\\ \bar{M} + L(\tau_3), & \tau > \tau_3. \end{cases} \quad \text{(S.28)}$$

Therefore, we are left to get rid of the influence of τ_3, or rather, to show that the incremental of $L(\tau)$ in the duration $\Delta = \tau_3 - \tau_2$ neither rely on the selection of time instants nor on the solution itself.

Note from (S.26) that on $[\tau_2,\, \tau_3]$, $\frac{\mathrm{d}L}{\mathrm{d}\tau} \leq \frac{\mathrm{d}M}{\mathrm{d}\tau} + \zeta_2^2$. Integrating both sides of it over duration Δ and recalling $L(\tau_2) = \tilde{\theta}_3$ and the unform boundedness of ζ_2 on $[0,\, +\infty)$ (denote by ζ_2^* the uniform bound), we see

$$L(\tau_3) \;\leq\; \tilde{\theta}_3 + \bar{M} + \zeta_2^{*2},$$

a constant independent of time instants and the solution itself. Thus, by (S.28) we conclude the sufficiency holds, i.e., $L(\tau)$ is uniformly bounded on $[0,\, +\infty)$ for uniformly bounded $M(\tau)$.

The proof is completed. $\square$

V. Equivalent Transformation

Consider the system with multiple unknown nonzero control coefficients g_i's:

$$\begin{cases} \dot{x}_i = g_i x_{i+1} + f_i(t, x), \quad i = 1, \ldots, n-1, \\ \dot{x}_n = g_n u + f_n(t, x), \\ y = x_1, \end{cases} \tag{S.29}$$

and the system with a single unknown nonzero control coefficient g:

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(t, x), \quad i = 1, \ldots, n-1, \\ \dot{x}_n = g u + f_n(t, x), \\ y = x_1. \end{cases} \tag{S.30}$$

The nonlinearities $f_i(\cdot)$'s in systems (S.29) and (S.30) satisfy Assumption 1, i.e.,

$$|f_i(t, x)| \;\leq\; \theta \phi(y) \sum_{j=1}^{i} \big(|x_j| + c|x_j|^{m_{i,j}}\big), \tag{S.31}$$

where $\phi(\cdot) \geq 1$ is a known smooth function, $m_{i,j}$'s are known constants satisfying $\frac{n-i}{n-j+1} \leq m_{i,j} < 1$, $j \leq i \leq n$, $\theta \geq 0$ is an unknown constant and $c \in [0,\, 1]$ is a constant.

We next show systems (S.29) can be equivalently transformed into system (S.30) in the output-feedback context. By performing

$$\xi = [\xi_1, \ldots, \xi_n]^{\mathrm{T}} = \mathbf{diag}\big\{1,\, g_1,\, g_1 g_2, \ldots, \Pi_{j=1}^{n-1} g_j\big\} x,$$

the unknown g_i's can be lumped together, and thus system (S.29) is transformed into:

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} + \phi_i(t, \xi), \quad i = 1, \ldots, n-1, \\ \dot{\xi}_n = g u + \phi_n(t, \xi), \\ y = \xi_1, \end{cases} \tag{S.32}$$

where unknown $g = \Pi_{j=1}^{n} g_j$ and the transformed nonlinearity $\phi_i(t, \xi) = (\Pi_{j=1}^{i} g_{j-1}) f_i(t, x)$ with $g_0 = 1$.

We can see system (S.30) and system (S.32) have the following identical features:

- They have the same lower-triangular structure.
- They have the same measurable output.
- Their 2nd up to n-th states are unmeasured.
- Their nonlinearities satisfy the same growth condition (S.31).

Therefore, we learn system (S.30) and system (S.32) are equivalent. Since system (S.32) has been shown to be the equivalent transformation of system (S.29), we know system (S.30) is also the equivalent transformation of system (S.29) in the output-feedback context.