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Adaptive Scaled Bipartite Consensus via Funnel Control

Linzhen Yu and Yungang Liu

***Abstract*—For consensus, one issue is that its various variants, such as group consensus, scaled consensus, and bipartite consensus, ought to be proposed as the expansion of research and applications. Another issue is that consensus should carry certain performance specifications, to meet some crucial demands, such as rapidity and safety. In this article, we investigate an integrated consensus (i.e., scaled plus bipartite consensus) with a prescribed convergence rate, for a class of uncertain nonlinear MASs. A funnel-based adaptive scheme is proposed for the scaled bipartite consensus. Specifically, a time-varying function is preselected to characterize the prescribed convergence rate. Utilizing this time-varying function and the relative states defined in the sense of the scaled bipartite consensus, a fully distributed protocol is designed. Therein, the funnel gains are critical ingredients to ensure the prescribed performance, since they would increase the control signal to be sufficiently large once the relative states approach the performance boundary. Particularly, the selected time-varying design function is added to act as a part of the control gain in the protocol, which is critical to forcing the ultimate convergence (to zero) of relative states. The proposed protocol is verified by a simulation example.**

***Index Terms*—Completely unknown nonlinearities, fully distributed protocol, funnel control, prescribed convergence rate, scaled bipartite consensus.**

I. Introduction

CONSENSUS, aiming at an agreement on the same value/trajectory, has been the primary objective for MASs [1], [2], [3], [4], [5], [6], [7]. With the expansion of research and applications, a lot of variants of consensus have been proposed. For example, group consensus means that agents are forced into several groups depending on the network while reaching intragroup agreements, via interactions not only within a group but between different groups [8]. Bipartite consensus under cooperative-antagonistic interactions refers to agents reaching a two-group agreement, with agreement variables in two groups owning the same size but opposite signs [9], [10]. Scaled consensus requires agents to agree on a set of proportional values/trajectories rather than a single one [11].

In this article, we consider an integrated consensus, i.e., scaled plus bipartite consensus, where the agreement variables are adjusted to be proportional and of opposite signs [12], [13]. More precisely, when cooperative-antagonistic interactions exist, it is desirable to assign different proportions (rather than the same amplitude) to agents in two opposite groups, reflecting different degrees of cooperation and competition. Such an integrated consensus, rather than any separate one, can be found in many actual scenarios/tasks. For example, people's opinions on an issue are not going to be completely opposite or in agreement. A massive search by multiple aircraft may occur in reverse regions. Robots on the same team will be assigned to different positions in robot competitions.

Apart from the integrated consensus, the prescribed performance is also taken into account in this article. This is a crucial step to further push the consensus forward to real-world applications, since synthesizing explicit and quantitative performance specifications helps to guarantee some vital actual demands, such as safety and rapidity [14], [15], [16], [17]. Remark that the scaled and bipartite consensus, ever since their emergence, have been separately studied, for various types of MASs (like integrator-type systems [9], [18], [19], [20], [21], general linear MASs [22], [23], and nonlinear ones [24], [25], [26]), over different kinds of graphs (like fixed undirected graphs [19] and switching graphs [20]), to name just a few. In contrast, there are fewer results concerning the scaled bipartite consensus [12], [13], and moreover, most of them exclude the prescribed performance. Recently, in works [27] and [28], the bipartite consensus has been achieved for uncertain nonlinear MASs, where the variables of interest are confined to evolving within the prescribed performance boundary while converging ultimately to residual sets. Whereas to the best of the authors' knowledge, no performance-related results on the scaled bipartite consensus have been reported.

Centering on the scaled bipartite consensus with a prescribed convergence rate, this article aims at a class of uncertain nonlinear MASs subject to unknown control coefficients and completely unknown nonlinearities. It is noted that raising the strength of system nonlinearities would bring increasing complications/difficulties in the control design and analysis. In works [19] and [26], bipartite consensus and scaled consensus were separately considered for MASs whose nonlinearities satisfy the Lipschitz-type condition. This condition enables the nonlinearities to transform into

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The authors are with the School of Control Science and Engineering, Shandong University, Jinan 250061, China (e-mail: yulz@mail.sdu.edu.cn; lygfr@sdu.edu.cn).

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relative-state-related forms so that the desired consensus can be achieved using only relative states. However, such nonlinearities are essentially homogeneous, which is rather restrictive for MASs. Work [25] addressed the bipartite consensus for a class of MASs where the nonlinearities are parameterized by certain known nonlinear functions and unknown constants. Such nonlinearities can be heterogeneous, but the known nonlinear functions therein are necessarily utilized to design the protocol, leading to the utilization of agents' absolute states. Furthermore, when the system nonlinearities are completely unknown, as in our result, the control design cannot acquire anything from nonlinearities. This, together with unknown control coefficients, directly results in insufficient model information, hindering the protocol design. Although the MASs are of rather coarse information, we still pursue the scaled bipartite consensus with the prescribed convergence rate. Moreover, an asymptotic steady state (i.e., the relative states asymptotically tending to zero), instead of a practical one as in [27] and [28], is to be performed.

Faced with prescribed performance requirements and insufficient model information, a powerful control strategy is demanded that can not only handle uncertainties/nonlinearities but guarantee a prescribed convergence rate. In this regard, the funnel control scheme comes as a good choice [14], [29], [30]. Specifically in this article, the first step is to preselect a suitable time-varying function characterizing the prescribed convergence rate and combine it with the relative states to design the funnel gains. Then, construct the delicate intermediate variables by appropriately matching the relative states and the funnel gains, based on which, a protocol in a fully distributed manner is designed. In particular, the preselected time-varying design function is introduced in the protocol to be a part of the control gain [see (2)], which potentially enhances the control effect, facilitating an asymptotic steady state. Additionally, as for the performance analysis, a critical task is to show the boundedness of the funnel gains. For this, we elaborately construct the Lyapunov function candidate by taking into account the properties of the directed signed graph. Integrating the boundedness of the funnel gains and the form of the intermediate variables, it is shown that the desired consensus objective is achieved.

The features/merits of this article are emphasized as follows.

1) The uncertain nonlinear MASs permit nonidentical unknown control coefficients and unknown nonlinearities with nonparametric uncertainties. Such heterogeneous MASs are challenging to achieve the desired consensus.
2) Scaled plus bipartite consensus, instead of simple consensus, is pursued. More than that, the consensus is achieved with the prescribed convergence rate and with the ultimate convergence to zero rather than a residual set.
3) The advanced funnel control approach is leveraged to design the protocol which only depends on relative states and which does not utilize any global graph information, despite the coarse model information of MASs.

The remainder of this article is organized as follows. Section II collects some preliminaries, including some notations, graph theory, and useful lemmas. Section III formulates the uncertain nonlinear MASs and the desired objective, and moreover, the significance of the considered problem is discussed. Section IV presents the design and analysis processes of the desired performance-prescribed scaled bipartite consensus. In Section V, the theoretical results are applied to a simulation example to verify the effectiveness. Section VI provides some concluding remarks.

II. Preliminaries

Notations: Let $\mathbf{R}$ and $\mathbf{R}_{\geq t_0}$ represent the sets of real numbers and real numbers not less than t_0, respectively. Denote the real N-dimensional Euclidean space by $\mathbf{R}_N$. Denote the N-dimensional column vector with all entries being 1 by $\mathbf{1}_N$. Let $\otimes$ denote the standard Kronecker product. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimal and maximal eigenvalues of the matrix A, respectively. Let $\text{sgn}(b)$ denote the sign of constant b.

Graph Theory: In MASs, the communication topology which is described as graph $\mathcal{G}$ is used to characterize information interactions between agents. A signed graph is denoted by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, respectively, denoting the sets of nodes (agents) and edges (information flows). An edge $(j, i) \in \mathcal{E}$ means that there is a directed path from node i to node j, and then j is called a neighbor of i. Matrix $A = [a_{ij}]_{N\times N}$, which satisfies that $a_{ij} \neq 0$, when $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, when $(j, i) \notin \mathcal{E}$, is the adjacency matrix. In particular, it is assumed that self-loops are not allowed, i.e., $a_{ii} = 0$. The associated Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}]_{N\times N}$ is formed as follows: $\mathcal{L}_{ij} = -a_{ij}$, for any $i \neq j$ and $\mathcal{L}_{ii} = \sum_{j\neq i} |a_{ij}|$.

A directed graph is said to contain a spanning tree implies that there is at least one node with directed paths to all of the other nodes. A signed graph is structurally balanced if there is a bipartition satisfying $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $a_{ij} \geq 0$ when nodes i and j are in the same partition and $a_{ij} \leq 0$ when i and j belong to different partitions. For a structurally balanced signed graph, the signature matrix $\Theta = \mathbf{diag}\{\theta_1, \ldots, \theta_N\}$ is defined with $\theta_i = 1\ \forall i \in \mathcal{V}_1$ and $\theta_i = -1\ \forall i \in \mathcal{V}_2$.

Furthermore, the augmented graph $\bar{\mathcal{G}} = \{\bar{\mathcal{E}}, \bar{\mathcal{V}}\}$ with $\bar{\mathcal{V}} = \{0, 1, \ldots, N\}$ and $\bar{\mathcal{E}} = \mathcal{E} \cup \{(0, i) : b_i \neq 0\} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$, characterizes the interaction between all $N+1$ agents. Therein, $b_i > 0$ implies cooperative relation while $b_i < 0$ implies competitive relation.

Technical Lemma: For a directed signed graph $\bar{\mathcal{G}}$, the following lemma is given, being helpful for the later consensus analysis. Their proofs can be found in works [31] and [32] and hence are omitted here.

Lemma 1: If the directed signed graph $\bar{\mathcal{G}}$ contains a spanning tree where the root is the leader and the subgraph $\mathcal{G}$ is structurally balanced, then the following statements hold.

1) The matrix $\mathcal{L}_1 = \mathcal{L} + \mathcal{B}$ is nonsingular and all its eigenvalues have positive real parts, where $\mathcal{B} = \mathbf{diag}\{|b_1|, \ldots, |b_N|\}$.
2) There is a positive diagonal matrix $P = \mathbf{diag}\{p_1, \ldots, p_N\}$ such that $Q = P\mathcal{L}_1 + \mathcal{L}_1^T P$ is positive definite.

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III. Problem Formulation

A. System Models

Consider a complex network composed of N followers and one leader. Therein, the involved ith follower, $i = 1, \dots, N$, has the following second-order uncertain nonlinear dynamics:

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} \\ \dot{x}_{i,2} = g_i u_i + f_i\big(t, x_{i,[2]}\big) \end{cases} \tag{1}$$

where $x_{i,[2]} = [x_{i,1}, x_{i,2}]^T \in \mathbf{R}_2$ is the system state vector of the ith follower, with $x_{i,[2]}(t_0) = x_{i0,[2]} = [x_{i,1}(t_0), x_{i,2}(t_0)]^T$; $u_i \in \mathbf{R}$ is the control input; the unknown constant g_i is termed the *control coefficient* whose sign is known; $f_i(\cdot)$ is called the *system nonlinearity* which is continuous and locally Lipschitz in its first and second arguments, respectively. Moreover, the leader state $x_0(t)$ is described as a continuously differentiable time-varying signal.

In what follows, two mild assumptions are imposed on the agents' dynamics.

Assumption 1: The unknown nonlinearities $f_i(t, x_{i,[2]})$'s are locally bounded in $x_{i,[2]}$ uniformly in t, which means

$$\big|f_i\big(t, x_{i,[2]}\big)\big| < \bar{f}_i\big(x_{i,[2]}\big), \quad i = 1, \dots, N$$

with $\bar{f}_i(x_{i,[2]})$'s being some unknown non-negative continuous functions.

Assumption 2: The leader state $x_0(t)$ is bounded and there is

$$\sup_{t \ge t_0}(|\dot{x}_0(t)| + |\ddot{x}_0(t)|) \ \le\ M$$

for an unknown constant $M > 0$.

Remark 1: Assumption 1 stresses the local boundedness of system nonlinearities with respect to each bounded set of $x_{i,[2]}$, which will be only utilized in the consensus analysis. That is, $f_i(\cdot)$ cannot provide anything for the protocol design. In addition, the lipschitzness satisfied by $f_i(\cdot)$ in this article is local and no global Lipschitz-type condition [19], [26] and parameterized form [25] is required. In this sense, the unknown system nonlinearities in this article would cover more types of nonlinear functions. Assumption 2 gives the boundedness of the leader trajectory and that of its first and second derivatives, which facilitates the exclusion of the finite-time escape of the closed-loop system solutions.

The considered $N+1$ agents interact with each other via the communication topology which is characterized by a signed graph $\bar{\mathcal{G}}$, and particularly the communication between followers is modeled by the subgraph $\mathcal{G}$. In this article, concerning the communication topology, we have the following typical assumption.

Assumption 3: The signed graph $\bar{\mathcal{G}}$ contains a directed spanning tree where the leader acts as the root and moreover, $\mathcal{G}$ is structurally balanced.

It is noted that, as revealed in the classical results [1], [9], the connectivity and structural balance required in Assumption 3 are necessary conditions to achieve consensus in the bipartite sense. More than that, under Assumption 3, according to Lemma 1, $\mathcal{L}_1$ is nonsingular, and its eigenvalues all have positive real parts. This will be frequently used in Section IV.

B. Control Objective

For N followers described as (1) and the leader $x_0(t)$, under Assumptions 1–3, we intend to achieve the performance-prescribed scaled bipartite consensus (scaled consensus over the directed signed graph $\bar{\mathcal{G}}$). Specifically, by virtue of the funnel control strategy, design a fully distributed protocol

$$u_i \ = \ u_i\big(\xi_{i,[2]}, \psi(t)\big), \quad i = 1, \dots, N$$

where $\xi_{i,[2]} = [\xi_{i,1}, \xi_{i,2}]^T \in \mathbf{R}_2$ with $\xi_{i,1} = \sum_{j=1}^N |a_{ij}| \cdot (\alpha_i x_{i,1} - \alpha_j \mathrm{sgn}(a_{ij}) x_{j,1}) + |b_i|(\alpha_i x_{i,1} - \mathrm{sgn}(b_i) x_0)$ and $\xi_{i,2} = \sum_{j=1}^N |a_{ij}|(\alpha_i x_{i,2} - \alpha_j \mathrm{sgn}(a_{ij}) x_{j,2}) + |b_i|(\alpha_i x_{i,2} - \mathrm{sgn}(b_i)\dot{x}_0)$, such that the following statements hold.

1) For given initial value of followers $x(t_0) = [x_{10,[2]}^T, \dots, x_{N0,[2]}^T]^T \in \mathbf{R}_{2N}$ and that of leader $x_0(t_0)$, all the closed-loop system signals are bounded on $[t_0, +\infty)$.
2) The *relative states* $\xi_{i,1}$ and $\xi_{i,2}$ converge with a prescribed convergence rate $1/\psi(t)$, namely, there are $\sup_{t \ge t_0} \psi(t)|\xi_{i,1}| < +\infty$ and $\sup_{t \ge t_0} \psi(t)|\xi_{i,2}| < +\infty$.
3) The scaled bipartite consensus is achieved: for *consensus errors* $e_{i,1} = \alpha_i x_{i,1} - \theta_i x_0$ and $e_{i,2} = \alpha_i x_{i,2} - \theta_i \dot{x}_0$, $i = 1, \dots, N$, there are $\lim_{t \to +\infty} e_{i,1}(t) = 0$ and $\lim_{t \to +\infty} e_{i,2}(t) = 0$; namely, the ultimate convergence (to zero) as time tends to be infinity is guaranteed.

Therein, α_i's are positive scale constants and θ_i's are defined as in the graph theory of Section II. Moreover, the function $\psi(t)$ used to characterize the prescribed convergence rate is continuously differentiable on $[t_0, +\infty)$ and particularly satisfies.

i) $\psi(t) > 0$, for any $t \ge t_0$.
ii) $\lim_{t \to +\infty} \psi(t) = +\infty$.
iii) $\dot{\psi}(t) \le c\psi(t)$ with constant $c > 0$.

Remark 2: We would like to specify the properties of $\psi(t)$ as follows.

1) In the later results, $\psi(t_0)$ is chosen to meet the initial value condition $\psi(t_0)|z_{i,k}(t_0)| < 1$ (see later Proposition 1), where $z_{i,k}$, $i = 1, \dots, N$, $k = 1, 2$ are intermediate variables in the designed protocol (i.e., (2)). This helps to ensure the transient performance of the system.
2) Combined with $\sup_{t \ge t_0} \psi(t)|\xi_{i,k}| < +\infty$, $i = 1, \dots, N$, $k = 1, 2$, the property ii) of $\psi(t)$ ensures the ultimate convergence (to zero) of the consensus errors.
3) Property iii) means that the prescribed $\psi(t)$ grows at most with an exponential rate, which contributes to estimations in the later performance analysis.

In fact, there are many familiar functions which can be chosen as $\psi(t)$, such as $\psi(t) = e^{\varrho_1 t} - \varrho_2$ with $\varrho_1 > 0$, $0 \le \varrho_2 < 1$, and $c \ge (\varrho_1/(1 - \varrho_2))$; $\psi(t) = \ln(\varrho_1 t + 1) + \varrho_2$ with $\varrho_1 > 0$, $\varrho_2 > 0$ and $c \ge (\varrho_1/\varrho_2)$. Therein, ϱ_1 and ϱ_2 depend on the initial values of system states owing to the initial condition $\psi(t_0)|z_{i,k}(t_0)| < 1$, $i = 1, \dots, N$, $k = 1, 2$.

C. Significance

First, one can see from the control objective described in Section III-B that two common consensus objectives are contained. Specifically, when $\alpha_i = 1$ for $i = 1, \dots, N$, the desired control objective becomes bipartite consensus;

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when $\theta_i = 1$ for $i = 1, \ldots, N$, i.e., the communication graph is completely cooperative, the achieved consensus is scaled. Moreover, essentially different from the qualitative consensus (without computable boundedness and/or convergence rate) [1], [2], [3], explicit and quantitative performance specifications for safety and rapidity are considered in this article, which is a crucial step in pushing consensus forward to real-life applications.

Second, the scaled bipartite consensus finds applications in many real-life scenarios, such as opinions in the decision process and massive research of multirobots in reverse regions. More than that, the integrated consensus under investigation is enlightening in terms of multiple agents assisting a single agent to perform a task more effectively. Consider a scenario, for example, in which an air defense missile intercepts an enemy missile. In this context, it is not difficult to design a controller for an agent (the air defense missile) to practically track a time-varying signal (the enemy missile) with the aid of existing results [33], [34]. Take this agent as the active (i.e., $u_0 \neq 0$) leader whose control input is designed to satisfy Assumption 2 (introduced in Section III-A). Then, the ε-neighborhood resulted from the practical tracking can be filled up to a certain extent if we assign N followers to achieve the scaled bipartite consensus for assisting the interception of the leader, which improves the success rate of the interception. Furthermore, it is worth pointing out that with an active leader, to the best of the authors' knowledge, few existing results can achieve asymptotic consensus tracking via a continuous control strategy, while this issue can be included in our result.

IV. Scaled Bipartite Consensus

In this section, the distributed protocol design and performance analysis for the performance-prescribed scaled bipartite consensus are presented. Specifically, invoking the funnel control strategy, a fully distributed protocol is first designed. Under the protocol, the existence and uniqueness of the solution concerning the resulting closed-loop system are discussed. Then, two propositions are given, respectively, characterizing the dynamic behaviors of the intermediate variables $z_{i,k}$'s and unfolding the implication relation between the boundedness of $z_{i,k}$'s and that of $h_{i,k}$'s. Finally, the main result is summarized into a theorem.

For each follower, we design the following protocol based on the relative states:

$$\begin{cases} z_{i,1} = \xi_{i,1} \\ h_{i,1}\big(t, z_{i,1}\big) = \frac{1}{1-(\psi(t)z_{i,1})^2} \\ z_{i,2} = h_{i,1}\big(t, z_{i,1}\big)\big(\kappa_{i,1} z_{i,1} + \xi_{i,2}\big) \\ h_{i,2}\big(t, z_{i,2}\big) = \frac{1}{1-(\psi(t)z_{i,2})^2} \\ u_i \;=\; -\operatorname{sgn}(g_i)\psi(t) h_{i,1}\big(t, z_{i,1}\big) h_{i,2}\big(t, z_{i,2}\big) z_{i,2} \end{cases} \tag{2}$$

where $\kappa_{i,1}$ is a constant to be designed; $h_{i,1}(\cdot)$ and $h_{i,2}(\cdot)$ are referred to as the funnel gains. Throughout this article, we will denote them for brevity by $h_{i,k}$ or $h_{i,k}(t)$ (with $k = 1, 2$) if no confusion arises.

Remark 3: One can see from (2) that the signs of g_i's are critical ingredients for the protocol design in this article. When the signs are unknown, i.e., unknown control directions are considered, there have been numerous results for MASs where system nonlinearities either satisfy Lipschitz-type condition or are with parametric uncertainties [35], [36], [37]. But, when unknown system nonlinearities as in this article exist, no related work has been reported to deal with the aggregations of unknown control directions (caused by distributed interactions). In this context, a specialized treatment is needed, integrating advanced techniques and even a new perspective, which deserves further study.

Noting the definitions of consensus errors $e_{i,k}$'s in Section III-B, with (2), we have the error dynamics shown as

$$\begin{cases} \dot{e}_{i,1} = e_{i,2} \\ \dot{e}_{i,2} = -\psi(t)|g_i|\alpha_i h_{i,1} h_{i,2} z_{i,2} + \alpha_i f_i\big(t, x_{i,[2]}\big) - \theta_i \ddot{x}_0. \end{cases}$$

Then, by defining $e_1 = [e_{1,1}, \ldots, e_{N,1}]^T$, $e_2 = [e_{1,2}, \ldots, e_{N,2}]^T$, $H_1 = \mathbf{diag}\{h_{1,1}, \ldots, h_{N,1}\}$, $H_2 = \mathbf{diag}\{h_{1,2}, \ldots, h_{N,2}\}$, $A = \mathbf{diag}\{\alpha_1, \ldots, \alpha_N\}$, $G = \mathbf{diag}\{|g_1|, \ldots, |g_N|\}$, $z_2 = [z_{1,2}, \ldots, z_{N,2}]^T$, $F(t, x) = [f_1(\cdot), \ldots, f_N(\cdot)]^T$ and $\Theta = \mathbf{diag}\{\theta_1, \ldots, \theta_N\}$, the error dynamics can be written in a compact form

$$\begin{cases} \dot{e}_1 \;=\; e_2 \\ \dot{e}_2 \;=\; -\psi(t) A G H_1 H_2 z_2 + A F(t, x) - \Theta \mathbf{1}_N \otimes \ddot{x}_0. \end{cases} \tag{3}$$

Concerning the closed-loop system (3), the *existence and uniqueness of the solution* is first discussed. From the definitions of $\xi_{i,k}$'s and $e_{i,k}$'s, one can readily obtain $\xi_k = \mathcal{L}_1 e_k$, with $e_k = [e_{1,k}, \ldots, e_{N,k}]^T$, $\xi_k = [\xi_{1,k}, \ldots, \xi_{N,k}]^T$. In what follows, unless otherwise specified, $i = 1, \ldots, N$, and $k = 1, 2$. By the relationship $\xi_k = \mathcal{L}_1 e_k$, and the fact $\mathcal{L}_1$ being a nonsingular matrix whose eigenvalues have positive real parts (according to Assumption 3 and Lemma 1), we define the following two sets:

$$\begin{cases} \mathbf{D}_1 = \left\{(t, e_1) \in \mathbf{R}_{\geq t_0} \times \mathbf{R}_N \,\middle|\, \psi(t)|z_{i,1}| < 1,\ i = 1, \ldots, N\right\} \\ \mathbf{D}_2 = \left\{(t, e_1) \in \mathbf{D}_1, e_2 \in \mathbf{R}_N \,\middle|\, \psi(t)\big|z_{i,2}\big(h_{i,1}, \xi_{i,[2]}\big)\big| < 1 \right. \\ \qquad\qquad\qquad\qquad\qquad\qquad \left. i = 1, \ldots, N\right\}. \end{cases}$$

From this and the forms of funnel gains in (2), we learn that when the system states tend to the boundaries of $\mathbf{D}_1$ and $\mathbf{D}_2$, the corresponding funnel gains grow to be sufficiently large and take effect of compensating serious uncertainties. From the definition of $h_{i,1}(\cdot)$, one can derive the smoothness of $h_1(\cdot) = [h_{1,1}(\cdot), \ldots, h_{N,1}(\cdot)]^T$ on $\mathbf{D}_1$. With this and $z_{i,2} = h_{i,1}(\cdot)(\kappa_{i,1} z_{i,1} + \xi_{i,2})$ in (2), it follows that $z_2 = [z_{1,2}, \ldots, z_{N,2}]^T$ is smooth on $\mathbf{D}_2$. Moreover, for $i = 1, \ldots, N$, the nonlinearity $f_i(\cdot)$ is continuous and locally Lipschitz on $\mathbf{D}_2$, in t and $x_{i,[2]}$, respectively, and $x_0(t)$ is a continuous differentiable signal on $\mathbf{D}_2$. From these, it follows that the right-hand side of (3) is the Lipschitz continuous in $e = [e_{1,[2]}^T, \ldots, e_{N,[2]}^T]^T \in \mathbf{R}_{2N}$ and continuous in t on $\mathbf{D}_2$. Note that, for given initial value $e(t_0) \in \mathbf{R}_{2N}$ satisfying $\psi(t_0)|z_{i,k}| < 1$, one can observe $(t_0, e(t_0)) \in \mathbf{D}_2$. Then, by [38, Th. 3.1, p. 18], the closed-loop system has a unique solution $(t, e(t)) \in \mathbf{D}_2$ on a small time interval $[0, t_s)$, and furthermore, according to [38, Th. 2.1, p. 17], the solution can be extended

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to the maximal existence interval $[t_0, t_f)$ with $t_0 < t_f \leq +\infty$. It is noted that if $t_f < +\infty$, then the closed-loop system states would tend to the boundaries of $\mathbf{D}_1$ and $\mathbf{D}_2$ as $t \to t_f$, and if $t_f = +\infty$, all the closed-loop system states are well-defined on $[0, +\infty)$ with respect to sets $\mathbf{D}_1$ and $\mathbf{D}_2$.

In what follows, we give two important propositions which will be useful for the later proof of Theorem 1.

Proposition 1: Let $V(t, z_2) = (1/2)\sum_{i=1}^{N} g_i p_i \alpha_i \log h_{i,2}$ with g_i, p_i and α_i being defined in (1), Lemma 1 and the control objective, respectively. Suppose that $\psi(t_0)|z_{i,k}| < 1$, $i = 1, \ldots, N$, $k = 1, 2$. Then, there is a small positive constant δ such that

$$\dot{V} \leq -\frac{1}{2}\psi(t)\sum_{i=1}^{N}\delta|g_i|p_i\alpha_i h_{i,2}^2 + \psi(t)\rho(h_1, z, x) \tag{4}$$

where $\rho(\cdot)$ is an unknown continuous positive function.

The proof of Proposition 1 is somewhat involved and will be given later in the section.

Proposition 2: Suppose $z_{i,k}(t)$'s are bounded on $[t_0, t_f)$. Then, $\sup_{t\in[t_0,t_f)} h_{i,k}(t, z_{i,k}(t)) < +\infty$ holds.

Proof: At first, we prove that $h_{i,1}(t)$'s is bounded on $[t_0, t_f)$. Let $V_{i,1} = (1/2)(\psi(t)z_{i,1})^2$. Then, along the system trajectories, the time derivative of $V_{i,1}(\cdot)$ satisfies

$$\dot{V}_{i,1} = \psi(t)\dot{\psi}(t)z_{i,1}^2 + \psi^2(t)z_{i,1}\dot{z}_{i,1} \quad \forall t \in [t_0, t_f). \tag{5}$$

By (2), one can readily obtain that $\dot{z}_{i,1} = \xi_{i,2} = h_{i,1}^{-1}z_{i,2} - \kappa_{i,1}z_{i,1}$, which, combined with $\psi(t)|z_{i,k}| < 1$, $h_{i,1}^{-1} = 1 - 2V_{i,1}$ and $\dot{\psi}(t) \leq c\psi(t)$, makes (5) be

$$\dot{V}_{i,1} \leq -2\big(\kappa_{i,1} - c + 1\big)V_{i,1} + 1. \tag{6}$$

Since $\kappa_{i,1} > c + 1$, we know that when $V_{i,1}(\cdot) > 1/(2(\kappa_{i,1} - c + 1))$, there is $\dot{V}_{i,1} < 0$. This implies $V_{i,1}(\cdot) < \max\{V_{i,1}(t_0), 1/(2(\kappa_{i,1} - c + 1))\} < (1/2)$. By the relationship between $V_{i,1}(\cdot)$ and $h_{i,1}(t)$, one can derive the boundedness of $h_{i,1}(t)$ on $[t_0, t_f)$.

We next prove the boundedness of $h_{i,2}(t)$'s on $[t_0, t_f)$. Noting that $z_{i,k}(t)$'s and $h_{i,1}(t)$'s are bounded, for $t \in [t_0, t_f)$, we can easily derive the boundedness of $\xi_{i,k}$'s, as well as that of $e_{i,k}$'s and $x_{i,k}$'s, under Assumption 2. This, together with the continuity of $\rho(\cdot)$ in Proposition 1, implies that there is an unknown positive constant $\bar{M}$ such that $\rho(\cdot) < \bar{M}$. The existence of constant $\bar{M}$ benefits from the boundedness imposed on MASs by Assumptions 1 and 2.

More than that, from the relation $h_{i,2} > 1$ [by the definitions in (2)], it follows that:

$$h_{i,2}^2 > h_{i,2} > \log h_{i,2}. \tag{7}$$

With (7) and $\rho(\cdot) < \bar{M}$ in hand, by invoking (4), we have

$$\dot{V} \leq \psi(t)\big(-\sigma V + \bar{M}\big) \quad \forall t \in [t_0, t_f). \tag{8}$$

Observing (8), one can obtain $\dot{V} < 0$ when $V(\cdot) > (\bar{M}/\sigma)$. This implies

$$V(t) < \max\left\{\frac{\bar{M}}{\sigma}, V(t_0)\right\} < +\infty \quad \forall t \in [t_0, t_f)$$

which, together with the definition of $V(\cdot)$ in Proposition 1, directly implies the boundedness of $h_{i,2}$'s. ■

Remark 4: Under the funnel control scheme, proving the boundedness of funnel gains is critical to the control effectiveness, especially to ensuring the prescribed convergence rate. The boundedness of $h_{i,1}$'s is conducted by selecting "sub"-Lyapunov function $V_{i,1}(\cdot)$. But for the boundedness of $h_{i,2}$'s, the similar treatment would lead to heavy couplings of funnel gains between neighboring agents (i.e., $h_{i,2}$ and $h_{j,2}$), owing to the distributed interactions. As such, we resort to $V(\cdot)$ in a summation form (i.e., $\sum_{i=1}^{N}(\cdot)$ in Proposition 1) which permits us to utilize the graph theory for dealing with the coupling. Particularly, using the logarithm function in $V(\cdot)$, which matches with the form of funnel gains $h_{i,2}$'s, provides the delicate treatment on the related term of $z_{i,2}\dot{z}_{i,2}$ [see the later treatment of (17)]. This helps to obtain (4) which is a prerequisite for the boundedness proof of $h_{i,k}$'s in Proposition 2. Moreover, the property of $y > \log y$ when $y > 1$ is helpful for the wanted form of $\dot{V}(t, z_2)$.

On the basis of the above two important propositions, we are ready to summarize the main theorem on the scaled bipartite consensus with the prescribed performance.

Theorem 1: Consider the uncertain nonlinear MASs composed by (1) and $x_0(t)$ under Assumptions 1–3. If $\psi(t)$ is preselected to satisfy $\psi(t_0)|z_{i,k}(t_0)| < 1$ for given initial values, the distributed funnel-based protocol (2) guarantees that, for given initial value $x(t_0) \in \mathbf{R}_{2N}$, all the closed-loop system signals (i.e., $x_{i,k}(t)$, $\xi_{i,k}(t)$ and $h_{i,k}(t)$ for $i = 1, \ldots, N$ and $k = 1, 2$) are bounded on $[t_0, +\infty)$. Moreover, the scaled bipartite consensus is achieved with $\xi_{i,k}$'s evolving with a prescribed convergence rate $1/\psi(t)$ and ultimately converging to zero.

Proof: Note that, for given initial value $x(t_0) \in \mathbf{R}_{2N}$, there is $\psi(t_0)|z_{i,k}(t_0)| < 1$, $i = 1, \ldots, N$, $k = 1, 2$. By the discussion on the existence and uniqueness of the solution, a unique solution exists on the maximal existence interval $[t_0, t_f)$. Then, one can readily obtain, for any $t \in [t_0, t_f)$

$$\psi(t)|z_{i,k}| < 1, \quad i = 1, \ldots, N, \ k = 1, 2. \tag{9}$$

In what follows, we first prove the boundedness of $z_{i,k}$'s and $x_{i,k}$'s on $[t_0, t_f)$. From (9) and $\psi(t) > 0$, for any $t > t_0$, it follows that $|z_{i,k}| < (1/\psi(t))$. Owing to $\lim_{t\to+\infty}\psi(t) = +\infty$, there must exist a finite time $\bar{t} \in (t_0, t_f)$ such that $\psi(t) > \epsilon$ for any $t \in (\bar{t}, t_f)$, with a positive constant ϵ. Then, one can obtain $|z_{i,k}(t)| < 1/\epsilon < +\infty$ $\forall t \in [\bar{t}, t_f)$. This, together with the selected initial system values, deduces the boundedness of $z_{i,k}(t)$'s on $[t_0, t_f)$ by the aid of the continuity of the closed-loop system. With the boundedness of $z_{i,q}$'s, noting Proposition 2, we can conclude that the funnel gains $h_{i,k}(t)$, $i = 1, \ldots, N$, $k = 1, 2$, are all bounded on $[t_0, t_f)$. Then, noting (2), the boundedness of $z_{i,k}$'s and $h_{i,k}$'s implies that of $\xi_{i,k}$'s. Keeping this in mind, by $\xi_k = \mathcal{L}_1 e_k$ and the nonsingularity of matrix $\mathcal{L}_1$, we can naturally conclude that $e_{i,k}$'s are also bounded on $[t_0, t_f)$. Then, by the definitions of $e_{i,k}$'s and Assumption 2, one can readily obtain the boundedness of $x_{i,k}$'s on $[t_0, t_f)$.

Moreover, from the boundedness of $\xi_{i,k}$'s and $h_{i,k}$'s, noting (2) and (9), we arrive at u_i, $i = 1, \ldots, N$, are bounded on $[t_0, t_f)$. Hence, no finite-time escape phenomenon occurs, i.e., $t_f = +\infty$. Furthermore, since (9) holds for $t_f = +\infty$, there

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must be $|z_{i,k}| < (1/\psi(t))$ on $[t_0, +\infty)$. Then, from the design of $z_{i,k}$'s in (2), it follows that:

$$\begin{cases} \xi_{i,1} &= z_{i,1} \\ \xi_{i,2} &= \left(h_{i,1}^{-1}(t)z_{i,2} - \kappa_{i,1}z_{i,1}\right) \end{cases} \tag{10}$$

which, together with $\psi(t) > 0$ and the boundedness of $h_{i,k}(t)$'s, directly implies

$$\sup_{t\geq t_0} \psi(t)|\xi_{i,k}| \;<\; +\infty, \quad i = 1, \ldots, N, \quad k = 1, 2.$$

Therefore, noting $\lim_{t\to+\infty} \psi(t) = +\infty$, $|\xi_{i,k}|$'s converge to zero at a rate of at least $1/\psi(t)$. In particular, by $\psi(t)|z_{i,1}(t)| < 1$ and $\xi_{i,1} = z_{i,1}$, we know that $\xi_{i,1}$ always evolves within the prescribed performance boundary described by $1/\psi(t)$, for any $t \in [t_0, +\infty)$. From the definitions of $e_{i,k}$'s, it follows that the desired scaled bipartite consensus is achieved. ■

It is noted that by the aid of $\xi_k = \mathcal{L}_1 e_k$ and the fact that all the eigenvalues of the nonsingular $\mathcal{L}_1$ have positive real parts (under Assumption 3), we deduce the following relationship:

$$\sum_{i=1}^{N} \xi_{i,k}^2 = \xi_k^T \xi_k = e_k^T \mathcal{L}_1^T \mathcal{L}_1 e_k \geq \lambda_{\min}\big(\mathcal{L}_1^T \mathcal{L}_1\big) \sum_{i=1}^{N} e_{i,k}^2. \tag{11}$$

Moreover, when the relative states are forced to converge with the prescribed convergence rate $1/\psi(t)$, one can obtain $\sup_{t\geq t_0}(\sum_{i=1}^{N} \psi^2(t) \cdot \xi_{i,k}^2) < +\infty$, by which and (11), we can derive $\sup_{t\geq t_0}(\sqrt{\lambda_{\min}(\mathcal{L}_1^T \mathcal{L}_1)}\psi(t)e_{i,k}) < +\infty$. From this, it follows that when the global information (i.e., $\sqrt{\lambda_{\min}(\mathcal{L}_1^T \mathcal{L}_1)}$) is available, the convergence rate of consensus errors (unavailable for the protocol design in MASs) can also be prescribed, by prescribing the convergence rate of relative states.

Remark 5: Funnel control is capable of ensuring the prescribed performance boundary of the states that are used to construct the funnel gains (i.e., $h_{i,k}$'s in this article). In this sense, observing from the forms of $h_{i,k}$'s in (2), what we ensured is the prescribed performance boundary of intermediate variables $z_{i,k}$'s rather than that of relative states $\xi_{i,k}$'s. It should be pointed that the introduction of intermediate variables is common for the control design of second-order (or higher-order) MASs. Particularly, the delicate forms of $z_{i,k}$'s facilitate the implication between the performance boundaries of intermediate variables (i.e., $\psi(t)|z_{i,k}| < 1$) and the prescribed convergence rate of relative states (i.e., $\sup_{t\geq t_0} \psi(t)|\xi_{i,k}| < +\infty$).

Remark 6: As for the ultimate convergence (to zero) of $\xi_{i,k}$'s, the property $\lim_{t\to+\infty} \psi(t) = +\infty$ provides the powerful guarantee. However, in the existing related results, this property generally came with some restrictions on system models [39] or was even not permitted [15], [17]. In this article, by extra introducing the $\psi(t)$ into the protocol and delicately constructing the intermediate variables (see (2) in Section IV), the property $\lim_{t\to+\infty} \psi(t) = +\infty$ is permitted, without imposing other restrictions on the systems.

Proof of Proposition 1: By the definitions of $h_{i,2}$'s in (2), we have $\dot{h}_{i,2} = 2h_{i,2}^2\psi(t)z_{i,2}(\dot{\psi}(t)z_{i,2} + \psi(t)\dot{z}_{i,2})$ and $1 - (\psi(t)z_{i,2})^2 = h_{i,2}^{-1}$ (by $h_{i,2}(t) > 0$ on $[t_0, t_f)$). With these in hand, taking the time derivative of $V(t, z_2)$ along the system trajectories, one can readily obtain, on $[t_0, t_f)$

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} \frac{1}{2}|g_i|p_i\alpha_i h_{i,2}^{-1}(t)\dot{h}_{i,2}(t) \\ &= \sum_{i=1}^{N} |g_i|p_i\alpha_i h_{i,2}\psi(t)\dot{\psi}(t)z_{i,2}^2 \\ &\quad + \sum_{i=1}^{N} |g_i|p_i\alpha_i h_{i,2}\psi^2(t)z_{i,2}\dot{z}_{i,2}. \end{aligned} \tag{12}$$

First, by $\dot{\psi}(t) \leq c\psi(t)$ and completing the square, we have

$$\begin{aligned} &\sum_{i=1}^{N} |g_i|p_i\alpha_i h_{i,2}\psi(t)\text{ọ}t\psi(t)z_{i,2}^2 \\ &\leq \sum_{i=1}^{N} c|g_i|p_i\alpha_i h_{i,2}\psi^2(t)z_{i,2}^2 \\ &\leq \psi(t)\sum_{i=1}^{N} c|g_i|p_i\alpha_i h_{i,2}|z_{i,2}| \\ &= \psi(t)\left(\sum_{i=1}^{N} \frac{h_{i,2}^2}{2l} + \frac{l\rho_1(z_2)}{2}\right) \end{aligned} \tag{13}$$

with l being a positive constant determined later and $\rho_1(z_2) = \sum_{i=1}^{N}(c|g_i|p_i\alpha_i z_{i,2})^2$ being a non-negative continuous function.

Next, we would like to tackle the second term on the right-hand side of (12). Noting the definitions of $z_{i,2}$'s in (2), we have

$$\begin{aligned} \dot{z}_{i,2} &= \dot{h}_{i,1}\big(\kappa_{i,1}z_{i,1} + \xi_{i,2}\big) + h_{i,1}\big(\kappa_{i,1}\xi_{i,2} + \dot{\xi}_{i,2}\big) \\ &= \dot{h}_{i,1}h_{i,1}^{-1}z_{i,2} + \kappa_{i,1}z_{i,2} - \kappa_{i,1}^2 h_{i,1}z_{i,1} + h_{i,1}\dot{\xi}_{i,2}. \end{aligned} \tag{14}$$

Therein, the term $\dot{h}_{i,1}$ can be calculated as follows, by combining with $\xi_{i,2} = h_{i,1}^{-1}z_{i,2} - \kappa_{i,1}z_{i,1}$ [according to (2)]:

$$\begin{aligned} \dot{h}_{i,1} &= 2h_{i,1}^2\psi(t)\dot{\psi}(t)z_{i,1}^2 + 2h_{i,1}\psi^2(t)z_{i,1}z_{i,2} \\ &\quad -2\kappa_{i,1}h_{i,1}^2\psi^2(t)z_{i,1}^2 \end{aligned}$$

with which and $\psi(t)|z_{i,2}| < 1$ for any $t \in [t_0, t_f)$, we have

$$\begin{aligned} &\sum_{i=1}^{N} |g_i|p_i\alpha_i h_{i,2}\psi^2(t)z_{i,2}^2\dot{h}_{i,1}h_{i,1}^{-1} \\ &\leq \sum_{i=1}^{N} 2|g_i|p_i\alpha_i h_{i,2}\psi|z_{i,2}|\big((c + \kappa_{i,1})h_{i,1} + 1\big) \\ &\leq \psi(t)\left(\sum_{i=1}^{N} \frac{h_{i,2}^2}{2l} + \frac{l}{2}\rho_2(z_2, h_1)\right) \end{aligned} \tag{15}$$

where the positive continuous function $\rho_2(z_2, h_1) = \sum_{i=1}^{N}(2g_i p_i\alpha_i z_{i,2}((c+\kappa_{i,1})h_{i,1}+1))^2$. Similar to (13), one can obtain

$$\begin{cases} \sum_{i=1}^{N} |g_i|p_i\alpha_i h_{i,2}\kappa_{i,1}\psi^2(t)z_{i,2}^2 \\ \qquad\qquad \leq \psi(t)\left(\sum_{i=1}^{N} \frac{h_{i,2}^2}{2l} + \frac{l\rho_3(z_2)}{2}\right), \\ \sum_{i=1}^{N} |g_i|p_i\alpha_i h_{i,2}\kappa_{i,1}^2 h_{i,1}\psi^2(t)z_{i,1}z_{i,2} \\ \qquad\qquad \leq \psi(t)\left(\sum_{i=1}^{N} \frac{h_{i,2}^2}{2l} + \frac{l\rho_4(z_2,h_1)}{2}\right) \end{cases} \tag{16}$$

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with $\rho_3(\cdot) = \sum_{i=1}^{N}(g_i p_i \alpha_i \kappa_{i,1} z_{i,2})^2$ and $\rho_4(\cdot) = \sum_{i=1}^{N}(g_i p_i \alpha_i \kappa_{i,1}^2 h_{i,1} z_{i,2})^2$ being non-negative continuous functions.

Moreover, noting (12) and the last term of (14), as well as $\xi_k = \mathcal{L}_1 e_k$, the term $\sum_{i=1}^{N}|g_i|p_i\alpha_i h_{i,2}h_{i,1}\psi^2(t)z_{i,2}\dot{\xi}_{i,2}$ can be written in a compact form, and specifically invoking (3), we have

$$\begin{aligned}&\sum_{i=1}^{N}|g_i|p_i\alpha_i h_{i,2}h_{i,1}\psi^2(t)z_{i,2}\dot{\xi}_{i,2}\\&= \psi^2(t)z_2^T GAH_1H_2P\mathcal{L}_1\dot{e}_2\\&= -\psi^3(t)z_2^T GAH_1H_2P\mathcal{L}_1H_2H_1AGz_2 + \psi^2(t)z_2^T GAH_1\\&\quad\cdot H_2P\mathcal{L}_1(AF(t,x) - \Theta\mathbf{1}_N\otimes\ddot{x}_0).\end{aligned}$$

By $z_2^T GAH_1H_2P\mathcal{L}_1H_2H_1AGz_2 = (1/2)z_2^T GAH_1H_2(P\mathcal{L}_1 + \mathcal{L}_1^T P)H_2H_1AGz_2$, and Lemma 1, one can readily obtain

$$\begin{aligned}&-\psi^3(t)z_2^T GAH_1H_2P\mathcal{L}_1H_2H_1AGz_2\\&= -\frac{1}{2}\psi^3(t)z_2^T GAH_1H_2QH_2H_1AGz_2\\&\le -\frac{1}{2}\psi(t)\lambda_{\min}(Q)\sum_{i=1}^{N}g_i^2\alpha_i^2h_{i,1}^2h_{i,2}^2\psi^2(t)z_{i,2}^2.\end{aligned}\tag{17}$$

This essentially benefits from the technically selected $V(\cdot)$ and the designed form of $h_{i,2}$'s.

Noting the definitions of $h_{i,2}$'s in (2) and $h_{i,2}(t) > 0$, we directly derive $\psi^2(t)z_{i,2}^2 = 1 - h_{i,2}^{-1}$, and hence we have

$$g_i^2\alpha_i^2h_{i,1}^2h_{i,2}^2\psi^2(t)z_{i,2}^2 = g_i^2\alpha_i^2h_{i,1}^2\left(h_{i,2}^2 - h_{i,2}\right).$$

This, together with $(\lambda_{\min}(Q))/2\sum_{i=1}^{N}g_i^2\alpha_i^2h_{i,1}^2h_{i,2} \le \sum_{i=1}^{N}(h_{i,2}^2)/(2l) + \sum_{i=1}^{N}(l/8)(\lambda_{\min}(Q)g_i^2\alpha_i^2h_{i,1}^2)^2$ by completing the square (with l being a positive constant determined later), directly implies that (17) can be further estimated as follows:

$$\begin{aligned}&-\psi^3(t)z_2^T GAH_1H_2P\mathcal{L}_1H_2H_1AGz_2\\&\le \psi(t)\left(\sum_{i=1}^{N}\left(\frac{1}{2l} - \frac{1}{2}\lambda_{\min}(Q)g_i^2\alpha_i^2h_{i,1}^2\right)h_{i,2}^2 + \frac{l\rho_5(h_1)}{2}\right)\end{aligned}\tag{18}$$

where $\rho_5(\cdot) = (1/4)\lambda_{\min}(Q)\sum_{i=1}^{N}(g_i^2\alpha_i^2h_{i,1}^2)^2$ is a positive continuous function.

In what follows, to complete the estimation for term $\sum_{i=1}^{N}|g_i|p_i\alpha_i h_{i,1}h_{i,2}\psi^2(t)z_{i,2}\dot{\xi}_{i,2}$, it remains to tackle $\psi^2(t)z_2^T GAH_1H_2P\mathcal{L}_1(AF(t,x) - \Theta\mathbf{1}_N\otimes\ddot{x}_0)$.

Noting $\psi^2(t)z_{i,2}^2 < 1$ and $h_{i,2} > 1$, one can readily obtain

$$\begin{aligned}\psi(t)\|z_2^T H_2\| &= \sqrt{\psi^2\left(z_{i,2}^2h_{i,2}^2 + \ldots + z_{N,2}^2h_{N,2}^2\right)}\\&\le \sqrt{\sum_{i=1}^{N}h_{i,2}^2} < \sum_{i=1}^{N}h_{i,2}.\end{aligned}$$

Moreover, by the feature of diagonal matrixes, there are $\|G\| = \max_{i=1,\ldots,N}\{|g_i|\}$, $\|P\| = \max_{i=1,\ldots,N}\{p_i\}$, $\|A\| = \max_{i=1,\ldots,N}\{\alpha_i\}$ and $\|\Theta\| = 1$. In addition, for the nonsingular matrix $\mathcal{L}_1$, $\|\mathcal{L}_1\|$ can be represented by $\sqrt{\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)}$. Keeping these in mind, by Assumption 1 and completing the square, one can obtain

$$\begin{aligned}&\psi^2(t)z_2^T GAH_1H_2P\mathcal{L}_1AF(t,x)\\&\le \psi(t)\Bigg(\sum_{i=1}^{N}\frac{h_{i,2}^2}{2l} + \frac{l}{2}\max_{m=1,\ldots,N}\left\{\left(g_k\alpha_k^2p_kh_{i,1}\right)^2\right\}\cdot\\&\qquad\lambda_{\max}\left(\mathcal{L}_1^T\mathcal{L}_1\right)\sum_{k=1}^{N}\bar{f}_k^2\Bigg)\\&= \psi(t)\left(\sum_{i=1}^{N}\frac{h_i^2}{2l} + \frac{l\rho_6(x,h_1)}{2}\right)\end{aligned}\tag{19}$$

where $\rho_6(\cdot) = \max_{m=1,\ldots,N}\{(g_k\alpha_k^2p_kh_{i,1})^2\}\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)\cdot\sum_{k=1}^{N}\bar{f}_k^2(\cdot)$ is a positive continuous function.

Following with similar treatments and noting Assumption 2, we can also obtain:

$$\begin{aligned}&-\psi^2(t)z_2^T GAH_1H_2P\mathcal{L}_1\Theta\mathbf{1}_N\otimes\dot{x}_0)\\&\le \psi(t)\Bigg(\sum_{i=1}^{N}\frac{h_{i,2}^2}{2l} + \frac{l}{2}\max_{m=1,\ldots,N}\left\{\left(g_k\alpha_kp_kh_{i,1}\right)^2\right\}\\&\qquad\cdot\lambda_{\max}\left(\mathcal{L}_1^T\mathcal{L}_1\right)M^2\Bigg)\\&= \psi(t)\left(\sum_{i=1}^{N}\frac{h_i^2}{2l} + \frac{l\rho_7(h_1)}{2}\right)\end{aligned}\tag{20}$$

with $\rho_7(\cdot) = \max_{m=1,\ldots,N}\{(g_k\alpha_kp_kh_{i,1})^2\}\lambda_{\max}(\mathcal{L}_1^T\mathcal{L}_1)M^2$ being a positive continuous function.

Substituting (13), (15), (16) and (18)–(20) into (12), one can readily yield

$$\dot{V} \le \psi(t)\left(\sum_{i=1}^{N}\left(\frac{7}{2l} - \frac{1}{2}\lambda_{\min}(Q)g_i^2\alpha_i^2h_{i,1}^2\right)h_{i,2}^2 + \rho(h_1,z,x)\right)$$

where $\rho(\cdot) = (l/2)\sum_{r=1}^{7}(\rho_r(\cdot))$ is an unknown positive continuous function.

Then, noting $h_{i,k}(t) > 1$ and taking sufficiently large $l > 7/(\lambda_{\min}(Q)\underline{g}^2\underline{\alpha}^2)$ with $\underline{g} = \min_{i=1,\ldots,N}\{|g_i|\}$ and $\underline{\alpha} = \min_{i=1,\ldots,N}\{\alpha_i\}$, there is a small positive constant ε immediately such that (4) holds.

This completes the proof of Proposition 1. ■

V. Simulation Example

In this section, a numerical example is given for the effective verification of the proposed distributed control scheme. Consider the MASs composed by one leader and five followers, interacting with each other by Fig. 1. The dynamics of the followers can be represented by (1) with the system nonlinearity $f_i(\cdot) = x_{i,1} + x_{i,2}$ and $m_i = i$. The leader trajectory is chosen as $x_0(t) = 10\sin(t)$. Obviously, Assumptions 1–3 required in the problem formulation are satisfied.

Let $t_0 = 0$, and the initial value is chosen as $x(0) = [x_{10,[2]}^T, \ldots, x_{50,[2]}^T]^T = [1, 5, 6, 3, 3, 1, 1, 4, 1, 5]^T$. The time-varying function is $\psi(t) = e^{0.2t} - 0.99$. Simple calculations verify the initial conditions $\psi(t_0)|z_{i,k}(t_0)| < 1$ for $i = 1, \ldots, 5$ and $k = 1, 2$. With the designed protocol (2),

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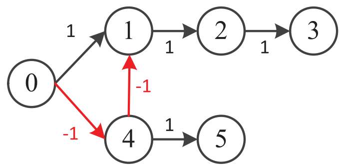

Fig. 1. Communication topology.

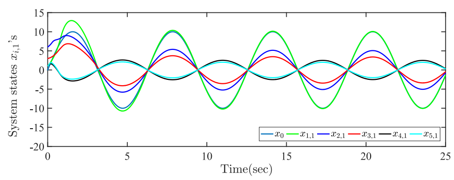

Fig. 2. Evolution of system states $x_{i,1}$'s.

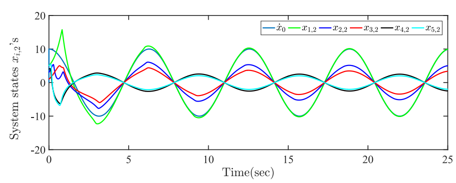

Fig. 3. Evolution of system states $x_{i,2}$'s.

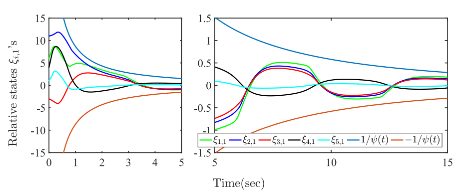

Fig. 4. Evolution of relative states $\xi_{i,1}$'s.

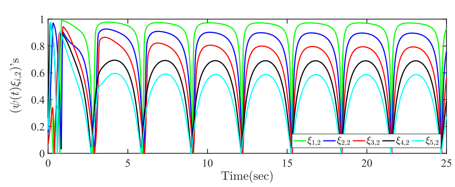

Fig. 5. Evolution of relative states $\xi_{i,2}$'s.

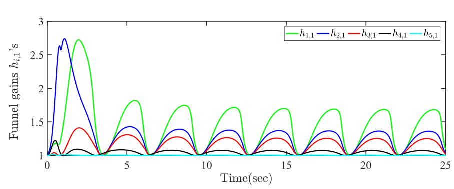

Fig. 6. Evolution of funnel gains $h_{i,1}$'s.

by simulation, we obtain the following figures (Figs. 2–8), exhibiting evolutions of the system state $x_{i,k}$, relative state $\xi_{i,k}$, funnel gain $h_{i,k}$ and control input u_i. In particular, Figs. 2 and 3 illustrate the scaled bipartite consensus is achieved, and meanwhile Figs. 4 and 5 show the prescribed performance (convergence rate) of the relative states by $|\xi_{i,1}| < (1/\psi(t))$ and $\sup_{t\geq 0} \psi(t)|\xi_{i,2}| < +\infty$.

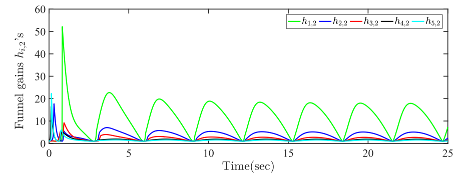

Fig. 7. Evolution of funnel gains $h_{i,2}$'s.

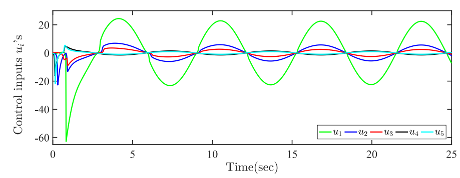

Fig. 8. Evolution of control inputs u_i's.

VI. Conclusion

This article has addressed the performance-prescribed scaled bipartite consensus problem for MASs subject to unknown control coefficients and completely unknown system nonlinearities. A funnel-based distributed control protocol in a fully distributed manner has been designed. It has been proved that the relative states defined in the sense of scaled bipartite consensus converge with a prescribed convergence rate and ultimately tend to be zero. The performance analysis has been done with the aid of the well-known Lyapunov method. Especially, constructing a suitable Lyapunov function candidate which involves the logarithm function as well as some parameters (i.e., θ_i and p_i) associated with the directed sign graph is rather critical to show the boundedness of funnel gains $h_{i,2}$'s. It is worth pointing out that the presented consensus scheme is based on the structural balance of the signed graph. Whether it is effective or how to design a new scheme, when the structural balance is violated, deserves further effort. Moreover, in the performance-prescribed context, optimal and fault-tolerant controls for uncertain nonlinear MASs or interconnected systems are also intriguing topics [7], [40].

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Linzhen Yu received the B.S. degree in automation from Jiangnan University, Wuxi, China, in 2018. She is currently pursuing the Ph.D. degree in control theory and control engineering with Shandong University, Jinan, China.

Her current research interests include multiagent systems, cooperative control, and adaptive control.

Yungang Liu received the Ph.D. degree in control theory and control engineering from Shanghai Jiao Tong University, Shanghai, China, in 2000.

He is currently a Changjiang Scholar Chair Professor with the School of Control Science and Engineering, Shandong University, Jinan, China, where he is also the Director of the Key Laboratory of Machine Intelligence and System Control, Ministry of Education of China, and the Institute of Artificial Intelligence and Systems and Control. He is the Director of the Technical Committee on Artificial Intelligence and Machine Vision, Shandong Institute of Electronics, Jinan; and the Director of the SDU-IBM Research Center on Big Data and Analytics. He is also the Vice Director of the Engineering Research Center of Intelligent Unmanned System, Ministry of Education of China, Jinan. His current research interests include stochastic control, nonlinear control design and system analysis, cooperative control, distributed parameter systems, adaptive control and applications, robots and motion control, and artificial intelligence.

Dr. Liu was a recipient of the National Outstanding Youth Science Foundation of China, the Special Government Allowances of State Council of China, and the Taishan Scholar Climbing Professor of Shandong Province of China. He was a recipient of the Guan Zhaozhi Award in Chinese Control Conference in 2004, the Second Prize of the National Natural Science Award of China in 2015, the Excellent Doctoral Dissertation Award of Chinese Association of Automation in 2018, and the Outstanding Graduate Student Instructor Awards of Shandong University and Shandong Province in 2019 and 2021.

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