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Global Adaptive Output-Feedback Tracking with Prescribed Performance for Uncertain Nonlinear Systems

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Abstract At present, one typical control strategy for guaranteeing transient and steady-state performance is funnel control and prescribed performance control. The strategy features completely discarding the system nonlinearities, even if they are completely known and available. Such an intrinsic feature requires the controller to produce a larger control effect to eliminate the negative impact caused by the high nonlinearities, leading to a conservative controller. In this paper, we fully take advantage of known information on nonlinearities in control design, instead of completely discarding it as done in funnel control. Particularly, we leverage adaptive techniques (i.e., high-gain dynamic compensation) to deal with unknown system nonlinearities. Meanwhile, we integrate the tools of output feedback, tracking control, and performance guarantee. As such, an adaptive output-feedback scheme is developed for global tracking with spatiotemporal performance specifications: arbitrarily given tracking accuracy and accuracy-ensured time. A simulation example supports the developed approach.

Keywords Uncertain nonlinear systems, output feedback, prescribed performance, unknown control direction, dynamic high gain, global tracking

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1 Introduction

For real plants, what the users/operators care about most tends to be the entire evolution of the concerned quantities during the whole system running, instead of only their steady state. This is due to poor transient behavior accounting for serious consequences, economically or safely. As such, it becomes an imperative and urgent task to shape the transient behavior as it is expected/prescribed to be in a safe system running. One typical control strategy for guaranteeing transient and steady-state performance is funnel control (FC) [1–4] as well as prescribed performance control (PPC) [5–8].

This paper attempts to settle global tracking[1)] via output feedback with prescribed performance for uncertain nonlinear systems in the context of *unknown control direction*. Notably, the unknown high nonlinearities are present in each signal channel, gravely perturbing system evolution and in turn entailing more powerful compensation. If the system in question degenerates to the Byrnes–Isidori canonical form (i.e., merely involving input matched nonlinearities) with *known* control directions, the *semi-global* tracking has already been solved in [6] by use of PPC proposed in [5] and has also been attacked in [3] by means of FC. In addition, semiglobal extensions have been performed for multiagent systems (MASs) with

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1) Throughout, global tracking is in the practical sense, i.e., for any initial value and any tracking accuracy λ, the tracking error can be steered to the λ-neighborhood of the origin.

measurement uncertainties [9] and stochastic MASs as well [10] by additionally using tuning functions. However, we are interested in a *global solution* instead of a *semiglobal one*, which naturally excludes the control strategies therein.

In contrast, a seminal work [2], also aiming for the special systems in the Byrnes–Isidori canonical form, utilized FC to realize global tracking. As a typical feature of FC, no information on system nonlinearities was used for control design. This feature is quite appealing when little or no information is known on the nonlinearities [2–4]. However, this, in turn, makes it hard for the available information on the nonlinearities to be used in FC. If there was available information on the nonlinearities, it would be reasonable and anticipated to fully exploit the information in controller design. For instance, nonlinearity $f(x)$ satisfies $|f(x)| \leqslant \theta\bar{f}(y)$ with known $\bar{f}(\cdot)$. Using the known $\bar{f}(y)$ in the controller, instead of discarding it as if $f(x)$ were completely unknown (as done in FC), could lead to less conservatism of the control effect.

On the other side, for a global goal [2,4], the performance boundaries with infinitely large initial values would potentially account for an excessively large bound of system states on initial intervals (i.e., the possible short-time inability of the controller). This is mainly because only when the states approach the boundary will the controller take drastic action to pull them back. However, when the states have not approached the arbitrarily large boundary on a small initial interval, the control action could be not strong enough and no other mechanism (like adaptive dynamic compensation) is used to help. Therefore, the unknown nonlinearities (not embodied in the controller) may rapidly "drive" the states to extremely large values.

With the above insights, we put forward a promising global output-feedback scheme by merging adaptive compensation. On the one hand, a dynamic high gain with tailored dynamics is leveraged to aid in governing the unknown high nonlinearities and mitigate the possible excessive largeness during the small initial interval. In the course, the known information of system nonlinearities is particularly used in control design, unlike FC where the nonlinearities are completely discarded at the risk of the potential short-time inability. By injecting the high gain into the observer, not only is the boundedness (instead of ISS) of scaled observer error directly obtained, but the design and analysis are largely simplified compared with [11] where two sets of Lyapunov functions are pursued to get around technical difficulties. Besides, we especially borrow the pseudosign and pseudo-dead-zone functions from [11,12], whose absolute values are sufficiently smooth. Their introduction reduces the use of completing the square in the control design and leads to a tighter and less conservative controller.

On the other hand, in this paper, the prescribed transient and steady-state performance manifests itself in two appealing specifications — *prescribed accuracy-ensured time* and *arbitrarily given tracking accuracy*. Confronted with the performance specifications, we resort to a time-varying performance boundary, inspired by the existing performance-guaranteed strategies [2,6,7,13]. The boundary itself does not enter into the controller, unlike in [5,6]; hence, it can have an infinitely large initial value, making the global goal feasible. In particular, multiple functions are pieced together to form the time-varying boundary, instead of merely one function being utilized, which allows the tracking error to have different evolution stages. Concretely, before reaching the prescribed accuracy, the tracking error is first steered to a relatively low level. Then after a pregiven amount of time (i.e., the low-level ensured stage), it enters the expected strip. Prescribing the two moderate tracking levels (or even more, if needed) not only guarantees the desired performance, but also gives consideration to the related cost (since simply pursuing a small tracking accuracy and/or small accuracy-ensured time could be at the cost of excessive control effect). To make the time-varying boundary shape the tracking error, a spatiotemporal performance function is exploited. By use of such a function, forcing the tracking error to evolve within the boundary naturally reduces to the boundedness of the spatiotemporal performance function. Subsequently, by integrating the spatiotemporal performance function into the Lyapunov function design, the control goal further reduces to the boundedness of all system signals, which is a less ambitious goal. It turns out, after concise Lyapunov analysis, the proposed adaptive controller achieves global tracking with guaranteed performance.

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2 Notations and Preliminaries

We assemble some notations used throughout this paper, and particularly introduce several important lemmas that will be used frequently in the later development.

For $x \in \mathbf{R}^n$, we use x_i to denote its i-th element, and let $x_{[i]} = [x_1, \ldots, x_i]^{\mathrm{T}}$. By $\mathbf{diag}\{x_1, \ldots, x_n\}$, we refer to the diagonal matrix with these x_i's on its principal diagonal. We write $\mathcal{C}^i$ for the set of all functions with continuous partial derivatives up to i-th order, $i \in \mathbf{N}$, and $\mathcal{C}^\infty$ for the set of all smooth functions.

We denote by $\mathbf{sign}(\cdot)$ the sign function. We use $N(\cdot)$ to denote the Nussbaum function with the following properties:

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k N(s)\,\mathrm{d}s = +\infty, \quad \liminf_{k\to\infty} \frac{1}{k}\int_0^k N(s)\,\mathrm{d}s = -\infty,$$

which is the sole way to the continuous solution for control problems with unknown control directions [12, 14, 15].

We now introduce five technical lemmas. The first two are drawn from [12], Lemmas 3 and 4 from [11], and the last one from [15].

Lemma 1 ([12]). There is an explicit function $P_n : [-1, 1] \to [-1, 1]$ with the following properties:

(i) $P_n(x)$ is $\mathcal{C}^n$ on $(-1, 1)$;

(ii) $P_n(0) = 0$, $\lim\limits_{x\to 1^-} P_n(x) = +1$ and $\lim\limits_{x\to -1^+} P_n(x) = -1$;

(iii) $\lim\limits_{x\to 1^-} P_n^{(i)}(x) = 0$ and $\lim\limits_{x\to -1^+} P_n^{(i)}(x) = 0$ for $i = 1, \ldots, n$, where $P_n^{(i)}(x)$ denotes the i-th derivative of $P_n(x)$.

Lemma 2 ([12]). The following defined pseudosign function $\mathrm{sgn}_{\mu,n}(\cdot)$ is $\mathcal{C}^n$ on $\mathbf{R}$ (indexed by parameter $\mu > 0$):

$$\mathrm{sgn}_{\mu,n}(x) = \begin{cases} \mathbf{sign}(x), & |x| \geqslant \mu, \\ P_n\left(\frac{x}{\mu}\right), & |x| < \mu, \end{cases}$$

which approaches the sign function as μ goes to zero.

We use Figure 1 to illustrate the sign function $\mathbf{sign}(x)$ and the pseudosign function $\mathrm{sgn}_{\mu,n}(x)$.

Figure 1 Sign function $\mathbf{sign}(x)$ and pseudosign function $\mathrm{sgn}_{\frac{1}{2},2}(x)$ with $P_2(x) = 12x^5 - 10x^3 + \frac{15}{4}x$, left and right, respectively.

Lemma 3 ([11]). There is an explicit function $Q_{\mu,n} : [-\mu, \mu] \to [0, \mu]$ with the following properties:

(i) $Q_{\mu,n}(x)$ is $\mathcal{C}^n$ on $(-\mu, \mu)$;

(ii) $\lim_{x\to\mu^-} Q_{\mu,n}(x) = \mu$ and $\lim_{x\to-\mu^+} Q_{\mu,n}(x) = 0$;

(iii) $\lim_{x\to\mu^-} Q_{\mu,n}^{(1)}(x) = 1$ and $\lim_{x\to\mu^-} Q_{\mu,n}^{(i)}(x) = 0$, $i = 2, \ldots, n$, and $\lim_{x\to-\mu^+} Q_{\mu,n}^{(j)}(x) = 0$, $j = 1, \ldots, n$.

Lemma 4 ([11]). The following defined pseudo-dead-zone function $D_{\mu,n}(\cdot)$ is $\mathcal{C}^n$ on $\mathbf{R}$ (indexed by parameter $\mu > 0$):

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$$D_{\mu,n}(x)=\begin{cases}(|x|-\mu)\mathbf{sign}(x), & |x|\geqslant\frac{3}{2}\mu,\\ Q_{\frac{\mu}{2},n}(|x|-\mu)\mathbf{sign}(x), & \frac{1}{2}\mu<|x|<\frac{3}{2}\mu,\\ 0, & |x|\leqslant\frac{1}{2}\mu,\end{cases}$$

which can be viewed as an approximation of the ideal dead zone function.

We use Figure 2 to illustrate the ideal dead zone and pseudo-dead-zone $D_{\mu,n}(x)$.

Figure 2 Ideal dead zone and pseudo-dead-zone $D_{1,2}(x)$, left and right, respectively. For $D_{1,2}(x)$, it is associated with $Q_{\frac{1}{2},2}(x)=-\frac{1}{2}x^4+\frac{3}{4}x^2+\frac{1}{2}x-\frac{3}{32}$; the blue curve segments in the first and third quadrants are generated by functions $-\frac{1}{2}x^4+2x^3-\frac{9}{4}x^2+x-\frac{5}{32}$ and $\frac{1}{2}x^4+2x^3+\frac{9}{4}x^2+x+\frac{5}{32}$, respectively.

Lemma 5 ([15])**.** Suppose that $V(\cdot)$ and $k(\cdot)$ are $\mathcal{C}^1$ functions defined on $[0,\,t_f)$, $0<t_f\leqslant+\infty$ and $V(\cdot)$ also is nonnegative. If for an even $\mathcal{C}^\infty$ Nussbaum function $N(\cdot)$, there is

$$V(t)\leqslant c_1+\int_0^t\Big(gN\big(k(s)\big)+c_2\Big)\dot{k}(s)\,\mathrm{d}s,\ \forall t\in[0,\,t_f),$$

then $k(t)$, $V(t)$ and $\int_0^t N\big(k(s)\big)\,\mathrm{d}k(s)$ are bounded on $[0,\,t_f)$, where c_1, c_2 and $g\neq0$ are constants.

3 Problem Formulation and Performance Boundary Establishment

3.1 System description

As is argued previously, transient behavior is as comparably important as steady-state behavior to guarantee the feasibility of controllers in practice. Typically, the exact duration that the tracking error spends in reaching the prescribed tracking accuracy acts as a critical specification for real plants. Therefore, we confine ourselves to *global* tracking with the prescribed performance, i.e., *arbitrarily given tracking accuracy* and *prescribed accuracy-ensured time*, for the following typical uncertain nonlinear system:

$$\begin{cases}\dot{x}_i=g_ix_{i+1}+f_i(t,x), & i=1,\dots,n-1,\\ \dot{x}_n=g_nu+f_n(t,x),\\ y=x_1,\end{cases}\tag{1}$$

where $x=[x_1,\dots,x_n]^{\mathrm{T}}\in\mathbf{R}^n$ is the system state vector with the initial value $x(0)=x_0$; $u\in\mathbf{R}$ and $y\in\mathbf{R}$ are the control input and measurable output of the system, respectively; $f_i(t,x)$'s are unknown nonlinearities, piecewise continuous in t and locally Lipschitz in x; g_i's are unknown nonzero constants, called control coefficients of the system.

Remark 1. Throughout, we limit ourselves to the context of *unknown control directions*. Such a severe uncertainty prevents system nonlinearities from essentially depending on unmeasured states [11, 16–19]. In fact, even for FC, the completely unknown nonlinearities have to be output-dependent in essence [2, 4].

Below we provide a new strategy to show how to make full use of the available information on nonlinearities in control design when pursuing a global goal and performance guarantee.

Assumption 1. For the unknown nonlinearities $f_i(t,x)$'s, there are known smooth nonnegative functions $\bar{f}_i(y)$'s and an unknown nonnegative constant θ, such that

$$|f_i(t,x)|\ \leqslant\ \theta\bar{f}_i(y),\quad i=1,\dots,n.$$

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Assumption 2. The reference signal y_r, only of known current value, is continuously differentiable. Moreover, there is an unknown constant $M_r \geqslant 0$ such that

$$\sup_{t\geqslant 0}(|y_r(t)| + |\dot{y}_r(t)|) \ \leqslant \ M_r.$$

Assumption 1 depicts an underlying nonlinear growth limitation that $f_i(\cdot)$'s should fulfill in the context of unknown control directions and output feedback [11]. As detailed in Remark 1, the nonlinearity growth is required to be output-dependent only, instead of unmeasured-state dependent, due to the unknown control directions. Overcoming the requirement calls for breakthroughs in control methodologies. On the other hand, many real plants can be simplified as special cases of system (1) under Assumption 1, such as controlled pendulum, Chua's circuit and ship steering [20–22].

Assumption 2 is a mild one for global tracking in the practical sense, under which merely some crude dynamic knowledge on y_r is required in addition to its current value. This assumption is in line with a mass of actual situations, such as path planning with autonomy and tracking the targets with maneuverability [20, 21], in which cases only the current value of the path/target y_r can be measured by sensors.

For all-around inspections of the control problem, the assumptions are also expected to be compared with those in the typical works of performance guarantee [1–6]. In [1, 2, 4], unknown system nonlinearities, though written as functions of all system states, are essentially *output dependent only*, akin to Assumption 1. More importantly, for the available part of the nonlinearities (i.e., the $\bar{f}_i(\cdot)$), it cannot be used in the controller design in FC strategies [1, 2, 4], but rather is still treated as completely unknown. This would potentially lead to a conservative controller and also entail complicated analysis to disclose the system performance. Work [5] put itself in the *state-feedback* scenario and required *known control directions*. Therein the nonlinearities could be much higher with respect to system states, while for output-feedback, high nonlinearities on unmeasured system states can lead to no global continuous output-feedback control (see [23]). Yet if one does not pursue a global goal but rather a semiglobal one, system nonlinearities and unknown control coefficients as well can be further relaxed (see, e.g., [3, 6]).

3.2 Control objective and line of thought

In this paper, we are dedicated to designing an adaptive output-feedback controller to achieve global tracking with the prescribed performance for system (1) under Assumptions 1 and 2. Specifically, the controller shall guarantee that for any initial data,

(i) all signals of the closed-loop system are well-defined and bounded on $[0, +\infty)$;

(ii) tracking error $e_r = y - y_r$ fulfills the *prescribed spatiotemporal constraint*: for any given times $T_{\bar{\lambda}}$ and T_λ $(0 < T_{\bar{\lambda}} < T_\lambda)$, tracking error decreases to a *moderately low level* $\bar{\lambda}$ at $T_{\bar{\lambda}}$, while it reaches the *anticipated tracking accuracy* $\lambda\,(\lambda < \bar{\lambda})$ at T_λ. Furthermore, on $[0, +\infty)$, the tracking error evolves within a prescribed performance region.

Here, $[0, T_{\bar{\lambda}})$ denotes the *transient stage* and $[T_{\bar{\lambda}}, +\infty)$ the *steady-state stage*; $[T_{\bar{\lambda}}, T_\lambda)$ stands for the *low-level-ensured stage* and $[T_\lambda, +\infty)$ the *accuracy-ensured stage*; see Figure 3 below.

Apparently, boundary construction and nonlinearity compensation are two key parts to performance guarantee. Note that FC as well as PPC uses the boundaries to enforce the evolution of the tracking error and to suppress the effect of the completely unknown nonlinearities [2]. However, the *global goal* pursued in this paper denies the use of PPC [5, 6]. In FC, the pursuit of a global goal, asking for boundaries with infinitely large initial values, could cause a short-time inability of the controller to suppress the completely unknown nonlinearities. This is mainly because only when the states approach the boundary will the controller take drastic action to pull them back. However, when the states have not approached the arbitrarily large boundary on a small initial interval, the control action could be not strong enough. In this way, if no other mechanism (like dynamic compensation in adaptive control), as done in FC, the unknown nonlinearities could rapidly "drive" the states to extremely large values. This potentially accounts for an excessively large bound of system states on small initial intervals.

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In view of these, besides the necessary boundaries, a dynamic high gain $L(t)$ is particularly introduced to help compensate for the nonlinearity, especially during the small initial interval. This, together with the boundaries, could better govern the error evolution and bring us some potential benefits on the boundedness of the systems.

Therefore, for the control problem, we will exploit the performance-guaranteed strategies and the adaptive compensation:

- The crucial performance boundary is expected to cover arbitrary initial data for the global goal, like FC while unlike PPC.
- We want to employ the available information of the system (i.e., the known $\bar{f}_i(\cdot)$'s in Assumption 1) by resorting to dynamic compensation, instead of discarding it as if the system nonlinearities were totally unknown.
- During the control design, we borrow the pseudo-dead-zone function $D_{\mu,n}(\cdot)$ and pseudosign function $\mathrm{sgn}_{\mu,n}(\cdot)$ from [11, 12] to reduce the use of completing the square for a tighter and less conservative controller.

3.3 Typicalness of the system model

System (1), subject to Assumption 1 and unknown control directions, is typical among its various variants in the output-feedback setting [11, 14, 15, 24–29]. To see its typicalness, we would like to exemplify its two extensions.

With the advanced techniques developed in the related works, the extra ingredients in the extensions would only bring about technical complexity, compared with system (1). Therefore, to highlight our core contributions in achieving *global output-feedback tracking* with *prescribed performance*, we may as well reduce the two extensions to the "core system", i.e., system (1).

Besides the theoretical typicalness, system (1) can also be utilized to model many real plants. Prominently, observed from system (1), the control coefficients that are unknown both in sign and magnitude act directly upon the (virtual) controls, rendering impossible any precise control effect. It thus could significantly weaken, or even worse, reverse the control effect, making things rather difficult in essence to solve. Therefore, by admitting the completely unknown g_i's, more serious faults/inaccuracies in an actuator (or equivalently in system output, e.g., $y = \theta x_1$) are encompassed, while substantial difficulties inevitably arise.

- *Accommodate unmodeled dynamics that are IS(p)S with respect to output y only:*

$$\begin{cases} \dot{\eta} = f_0(\eta, y), \\ \dot{x}_i = g_i x_{i+1} + f_i(t, \eta, x), \quad i = 1, \dots, n-1, \\ \dot{x}_n = g_n u + f_n(t, \eta, x), \\ y = x_1. \end{cases} \tag{2}$$

In this case, the negative influence of inverse dynamics η on x-subsystem manifests in nonlinearities $f_i(\cdot)$'s, according to which Assumption 1 should be adjusted to

$$|f_i(t, \eta, x)| \leqslant \theta \bar{f}_i(y) + \theta \psi_i(\|\eta\|), \quad i = 1, \dots, n,$$

with known $\mathcal{C}^\infty$ functions $\psi_i(\cdot) \geqslant 0$. With IS(p)S properties, the inverse dynamics in system (2) can be dealt with by adopting advanced techniques in [29, 30]. Literally, merely technical complexity in the treatment of $\psi_i(\cdot)$'s would emerge.

- *Admit bounded additive noises in each signal channel and measurement uncertainties:*

$$\begin{cases} \dot{x}_i = g_i x_{i+1} + f_i(x) + \omega_i(t), \quad i = 1, \dots, n-1, \\ \dot{x}_n = g_n u + f_n(x) + \omega_n(t), \\ y = \theta x_1. \end{cases} \tag{3}$$

For such a system, perform transformation $\xi = \theta x$ and let $\tilde{f}_i(t, x) = \theta\big(f_i(x) + \omega_i(t)\big)$; then system (3) can be converted into system (1) directly and Assumption 1 holds with

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$$|\tilde{f}_i(t,x)| \leqslant \theta^* \hat{f}_i(y), \quad i=1,\ldots,n,$$

for some unknown $\theta^* > 0$ and known $\mathcal{C}^\infty$ functions $\hat{f}_i(y) \geqslant 0$.

3.4 Crucial performance boundary

Performance boundary characterizes/confines, to a certain extent, the overall underlying evolution of some system states. Not only this, but it also virtually plays a decisive role in whether a control problem can be achieved globally or semiglobally. Note that a satisfactory boundary shapes system behaviors well, transient and steady-state; thus, it would be instrumental in guaranteeing the concerned targets (e.g., convergence rate and tracking accuracy).

Since we are concerned with realising *global* tracking with prescribed performance in both *temporal* and *spatial* aspects, the expected performance boundary ought to ***i)*** cover all possible initial values of the tracking error at the initial time instant; and ***ii)*** decrease to any anticipated level at any prescribed time, as aforementioned in control objective ***(ii)***.

Motivated by performance-guaranteed controls [2, 3, 6, 7, 13], we construct the following performance boundary:

$$\mathcal{F}(s(t)) = \frac{s(t)}{\sqrt{1-s^2(t)}}, \quad t>0, \tag{4}$$

where the (piecewise) design function $s(t)$, helping boundary $\mathcal{F}(\cdot)$ fulfill items ***i)*** and ***ii)*** above, is picked satisfying:

(i) $s(t)$ belongs to $\mathcal{C}^n$ on $[0,+\infty)$, monotonically decreasing, $s(0)=1$ and $s(t)<1$, $t>0$;

(ii) $s(T_{\bar{\lambda}}) = \frac{\bar{\lambda}}{\sqrt{1+\bar{\lambda}}}$, $s(T_\lambda) = \frac{\lambda}{\sqrt{1+\lambda}}$ and $s(t)=s(T_\lambda)$, $t>T_\lambda$;

(iii) $s(t), \dot{s}(t), \ldots, s^{(n-1)}(t)$ are bounded and available for feedback, and $|\dot{s}(t)| \leqslant M_s$ for a known $M_s>0$.

We want to briefly explain the underlying attributes of the prescribed boundary $\mathcal{F}(\cdot)$ and how the global tracking performance is guaranteed through these boundary/function attributes:

- Observed from (4) and the properties of $s(t)$, boundary $\mathcal{F}(s(t))$ is an increasing function with respect to $s(t)$ and a decreasing one concerning t.
- At the initial time instant, by letting $s(0)=1$, it holds that $\mathcal{F}(s(0^+)) = \mathcal{F}(1^-) = +\infty$ (Figure 3), which naturally enables $e_r(0) < \mathcal{F}(s(0^+)) = +\infty$, $\forall e_r(0) \in \mathbf{R}$, i.e., the anticipated property ***i)*** of $\mathcal{F}(\cdot)$, in line with the global objective of the control problem.
- By requiring $s(t)$ to pass through the pregiven points $(T_{\bar{\lambda}}, s(T_{\bar{\lambda}}))$ and $(T_\lambda, s(T_\lambda))$, boundary $\mathcal{F}(s(t))$ shall reach low level $\bar{\lambda}$ and tracking accuracy λ at prescribed time instants, as argued below, which renders the fulfillment of the tracking performance given in control objective ***(ii)***.

Inspired by $D_{\mu,n}(\cdot)$ that is sufficiently smooth (see Lemma 4), we provide an explicit expression of the anticipated $s(t)$ that fulfills properties ***(i)*** and ***(ii)*** so that boundary $\mathcal{F}(s(t))$ characterizes different evolution stages of the tracking error for the later design development.

Since we expect the magnitude of $\mathcal{F}(s(t))$ decreases to a moderately low level $\bar{\lambda}$ at time instant $T_{\bar{\lambda}}$ and decreases to λ $(\leqslant \bar{\lambda})$ at T_λ $(> T_{\bar{\lambda}})$, we have

$$\mathcal{F}(s(T_{\bar{\lambda}})) = \frac{s(T_{\bar{\lambda}})}{\sqrt{1-s^2(T_{\bar{\lambda}})}} = \bar{\lambda}, \quad \mathcal{F}(s(T_\lambda)) = \frac{s(T_\lambda)}{\sqrt{1-s^2(T_\lambda)}} = \lambda,$$

which give $s(T_{\bar{\lambda}}) = \frac{\bar{\lambda}}{\sqrt{1+\bar{\lambda}}}$ and $s(T_\lambda) = \frac{\lambda}{\sqrt{1+\lambda}}$, in accordance with property ***(ii)*** of $s(t)$.

Now, select $\mathcal{C}^n$ piecewise function $s(t)$ as

$$s(t) = \begin{cases} -\frac{1-s(T_{\bar{\lambda}})}{T_{\bar{\lambda}}} t + 1, & 0 \leqslant t \leqslant T_{\bar{\lambda}}, \\ s(T_\lambda) + \overline{Q}_{\nu,n}(-t + \frac{1}{2}(T_{\bar{\lambda}} + T_\lambda)), & T_{\bar{\lambda}} < t < T_\lambda, \\ s(T_\lambda), & t \geqslant T_\lambda, \end{cases} \tag{5}$$

where $\overline{Q}_{\nu,n}(t)$, analogous to $Q_{\mu,n}(\cdot)$ in Lemma 3, is picked such that

- $\overline{Q}_{\nu,n}(t)$ is $\mathcal{C}^n$ on $(-\nu,\,\nu)$ with $\nu=\frac{1}{2}(T_\lambda-T_{\bar\lambda})$;
- $\lim\limits_{t\to\nu^-}\overline{Q}_{\nu,n}(t)=s(T_{\bar\lambda})-s(T_\lambda)$ and $\lim\limits_{s\to-\nu^+}\overline{Q}_{\nu,n}(t)=0$;
- $\lim\limits_{t\to\nu^-}\overline{Q}^{(1)}_{\nu,n}(t)=\frac{1-s(T_{\bar\lambda})}{T_{\bar\lambda}}$ and $\lim\limits_{t\to\nu^-}\overline{Q}^{(i)}_{\nu,n}(t)=0$, $i=2,\dots,n$, and $\lim\limits_{t\to-\nu^+}\overline{Q}^{(j)}_{\nu,n}(t)=0$, $j=1,\dots,n$.

Figure 3 illustrates the above-formulated performance boundary $\mathcal{F}(\cdot)$ as well as the associated design function $s(t)$.

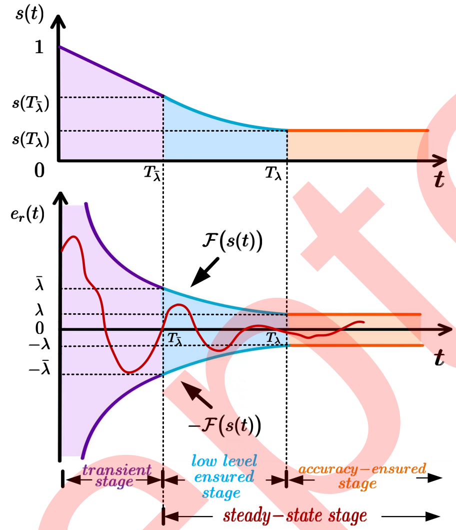

Figure 3 Evolution of the tracking error $e_r(t)$ with the prescribed performance.

4 Observer Design and Dynamic High Gain

In this section, we devote ourselves to working out a dynamic-high-gain based observer and deriving the crucial dynamics of the high gain. But before moving on, the following transformation needs to be performed on system (1) first, to gather unknown g_i's that are distributed in each signal channel:

$$\xi=[\xi_1,\dots,\xi_n]^{\mathrm{T}}=\mathbf{diag}\{1,\,g_1,\,g_1g_2,\dots,\Pi_{j=1}^{n-1}g_j\}x. \tag{6}$$

By this, system (1) is transformed into

$$\begin{cases}\dot\xi_i=\xi_{i+1}+\phi_i(t,\xi), & i=1,\dots,n-1,\\ \dot\xi_n=gu+\phi_n(t,\xi),\\ y=\xi_1,\end{cases} \tag{7}$$

where unknown $g=\Pi_{j=1}^n g_j$ and the transformed nonlinearity $\phi_i(t,\xi)=\Pi_{j=1}^i g_{j-1}f_i(t,x)|_{(6)}$ with $g_0=1$.

Accordingly, the system nonlinear growth given in Assumption 1 is transformed into

$$|\phi_i(t,\xi)|\leqslant\bar\theta\bar\phi_i(y), \tag{8}$$

with known smooth nonnegative functions $\bar\phi_i(\cdot)$'s and unknown positive constant $\bar\theta$.

Taking system (7) as a new starting point, we design the dynamic-high-gain observer as follows:

$$\begin{cases}\dot{\hat\xi}_i=\hat\xi_{i+1}-L^ia_i\hat\xi_1, & i=1,\dots,n-1,\\ \dot{\hat\xi}_n=u-L^na_n\hat\xi_1.\end{cases} \tag{9}$$

In (9), dynamic high gain L is generated by

$$\dot L=-\delta_1L^2+\delta_2L\big(1+y^2+\|\bar\phi(y)\|^2\big),\quad L(0)\geqslant1, \tag{10}$$

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where $\bar\phi(y)=\big[\bar\phi_1(y),\ldots,\bar\phi_n(y)\big]^{\rm T}$ and positive parameters δ_i's are such that $\delta_1\leqslant\frac{1}{2c_2}<\frac{1}{c_1}\leqslant\delta_2$ with c_i's satisfying (13) below.

We can see that dynamic high gain $L(t)$ with tailored dynamics (10) has three features corresponding to three indispensable roles in achieving the adaptive global tracking:

- $L(t)$ always satisfies $L(t)\geqslant 1$ to be a high gain, since $\delta_2>\delta_1$ and $L(0)\geqslant 1$. This allows $L(t)$ to be sufficiently large to capture system uncertainties.
- $L(t)$ is BIBS, i.e., it is bounded for bounded measurable output y. As such, the boundedness of $L(t)$ reduces to that of y, while the latter is implied by the achievement of the tracking goal.
- $L(t)$ has a tailored nonlinear part "$y^2+\|\bar\phi(y)\|^2$" in its dynamics, which makes it possible for $L(t)$ to counteract high nonlinearities originating in the control design.

Due to the unknown control coefficient g in (7), we choose observer errors $e_i=g\hat\xi_i-\xi_i,\ i=1,\ldots,n$, by which the following *control-free* error dynamics are straightforward to come by:

$$\begin{cases}\dot e_i\ =\ e_{i+1}-L^ia_ie_1-\phi_i(t,\xi)-L^ia_i\xi_1,\\ \dot e_n\ =\ -L^na_ne_1-\phi_n(t,\xi)-L^na_n\xi_1.\end{cases}\tag{11}$$

Let the scaled observer error be $\varepsilon_i=\frac{e_i}{L^i}$. It is then immediate from (11) to gain ($\varepsilon=[\varepsilon_1,\ldots,\varepsilon_n]^{\rm T}$)

$$\dot\varepsilon\ =\ LA_a\varepsilon-\Phi(t,\xi,L)-ay-D\varepsilon\frac{\dot L}{L},\tag{12}$$

with $A_a=\big[-a,\,[\mathbf{I}_{n-1},\mathbf{0}_{(n-1)\times 1}]^{\rm T}\big]$, $a=[a_1,\ldots,a_n]^{\rm T}$, $\Phi(\cdot)=[\frac{\phi_1(t,\xi)}{L},\ldots,\frac{\phi_n(t,\xi)}{L^n}]^{\rm T}$ and $D=\mathbf{diag}\{1,2,\ldots,n\}$.

Choose a_i's such that matrix A_a is Hurwitz and there are

$$\begin{cases}A_a^{\rm T}P+PA_a\ \leqslant -2\mathbf{I}_n,\\ c_1\mathbf{I}_n\ \leqslant\ PD+D^{\rm T}P\ \leqslant\ c_2\mathbf{I}_n,\end{cases}\tag{13}$$

for a symmetric positive definite matrix P, and positive constants c_1 and c_2 meeting $c_1<c_2$.

Remark 2. In the literature regarding observer design, observer states $\hat x_i$ are required to *reconstruct* system states x_i as truly as possible; thus, observer error is usually defined as $e_i=\hat x_i-x_i$. In contrast, observers for output-feedback control aim to *dynamically compensate* for the unmeasured states; thus, the selection of observer error has more flexibility, as long as it facilitates the output-feedback control design and analysis. Here, we pick $e_i=g\hat\xi_i-\xi_i$, rather than $e_i=\hat\xi_i-\xi_i$. By doing so, the observer error dynamics (11) are rendered *control-free* and *tractable*. The control-free property indicates certain separation between the observer and the controller, which is indispensable for the observer-based controller design.

For the crucial scaled error ε argued above, we would like to unfold its stability of boundedness type which is subtler and more tractable than a relation of the ISS type as in [11, 15, 17] in the context of practical tracking, due to the introduction of high gain L with delicately devised dynamics (10).

Proposition 1. Let $V_\varepsilon=\varepsilon^{\rm T}P\varepsilon$ with P satisfying (13). Along trajectories of (12) and (10), there is (for unknown $\theta_\varepsilon>0$)

$$\dot V_\varepsilon\ \leqslant\ -\frac{3L}{2}\|\varepsilon\|^2+\theta_\varepsilon,\tag{14}$$

which naturally gives the boundedness of ε.

Proof. Taking the time derivative of V_ε and invoking (12), we have

$$\dot V_\varepsilon\ =\ L\varepsilon^{\rm T}(A^{\rm T}P+PA)\varepsilon-2\varepsilon^{\rm T}P\Phi(t,\xi,L)-2\varepsilon^{\rm T}Pay-\varepsilon^{\rm T}(D^{\rm T}P+PD)\varepsilon\frac{\dot L}{L}.\tag{15}$$

By noting $\dot L$ in (10) and $L(t)\geqslant 1$ and applying (8), the indefinite terms in (15) satisfy

$$\begin{cases}2\varepsilon^{\rm T}P\Phi(t,\xi,L)\ \leqslant\ \|\varepsilon\|^2\|\bar\phi(y)\|^2+n\bar\theta^2\|P\|^2,\\ 2\varepsilon^{\rm T}Pay\ \leqslant\ \|\varepsilon\|^2y^2+\|Pa\|^2,\\ -\varepsilon^{\rm T}(D^{\rm T}P+PD)\varepsilon\frac{\dot L}{L}\ \leqslant\ \delta_1c_2L\|\varepsilon\|^2-\delta_2c_1\big(1+y^2+\|\bar\phi\|^2\big)\|\varepsilon\|^2.\end{cases}$$

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Putting the estimates into (15) and noting $\delta_1 \leqslant \frac{1}{2c_2}$ and $\delta_2 \geqslant \frac{1}{c_1}$ directly lead to (14) with $\theta_\varepsilon = n\bar{\theta}^2\|P\|^2 + \|Pa\|^2$.

Noting the definition of V_ε, we see a constant $\kappa > 0$ exists such that $\dot{V}_\varepsilon \leqslant -\kappa L V_\varepsilon + \theta_\varepsilon$. Solving it directly arrives at the boundedness of ε, whether high gain L is bounded or not. □

Remark 3. Observed from estimations above, dynamic high gain L lends itself well to the counteraction of nonlinearity "$\big(y^2 + \|\bar{\phi}(\cdot)\|^2\big)\|\varepsilon\|^2$". By injecting L into observer (9), the negative definite term with factor "$-\frac{\dot{L}}{L}$" would emerge, providing negative counterpart "$-\big(1 + y^2 + \|\bar{\phi}(\cdot)\|^2\big)\|\varepsilon\|^2$". This key term directly eliminates the unwanted positive counterpart and, in turn, helps the establishment of satisfactory tractable stability of the boundedness type rather than the ISS type.

5 Adaptive Output-feedback Controller Design

In this section, we devote ourselves to designing an adaptive controller via output feedback for system (7) to realize global tracking with prescribed performance.

Recall that anticipated tracking performance embodies largely into the behavior of tracking error e_r; thus, keeping e_r evolving within the designed performance boundary $\mathcal{F}(s(t))$ turns into our overarching task, as required in control objectives. To this end, we need to associate tracking error e_r with boundary $\mathcal{F}(s(t))$ by using special performance functions that could blow up when e_r approaches boundary $\mathcal{F}(s(t))$.

In this direction, we introduce the critical spatiotemporal performance function

$$h\big(s(t), e_r\big) = \frac{v(s(t), e_r)}{1 - v^2\big(s(t), e_r\big)}, \tag{16}$$

with

$$v(s(t), e_r) = \frac{w(e_r)}{s(t)}, \quad w(e_r) = \frac{e_r}{\sqrt{e_r^2 + 1}}. \tag{17}$$

Remark 4. The design of functions $h(\cdot)$, $v(\cdot)$ and $w(\cdot)$ in (16) and (17) is in effect objective-oriented. To understand the intuitions behind them, recall that the tracking error is expected to satisfy $\mathcal{F}(-s(t)) < e_r(t) < \mathcal{F}(s(t))$, which gives $-s(t) < \mathcal{F}^{-1}(e_r(t)) < s(t)$. Observe from (4) that $\mathcal{F}^{-1}(e_r) = \frac{e_r}{\sqrt{e_r^2+1}} =: w(e_r)$. Then, by using $s(t) > 0$ delineated in Section 3.4, there is $\frac{|\mathcal{F}^{-1}(e_r)|}{s(t)} < 1$. Therefore, we let $v(t, e_r) := \frac{\mathcal{F}^{-1}(e_r)}{s(t)} = \frac{w(e_r)}{s(t)}$, and the expected spatiotemporal performance function $h(\cdot)$, which is a function of $v(\cdot)$, should blow up as $|v(\cdot)|$ approaches 1. In view of these, we pick $h(\cdot)$ as in (16), by which ensuring e_r to evolve within performance boundary $\mathcal{F}(s(t))$ naturally comes down to checking the boundedness of $h(\cdot)$.

We now design the following adaptive output-feedback controller for system (7):

$$\begin{cases} u = \alpha_n(S(t), k, L, y, y_r, \hat{\xi}_{[n]}), \\ \dot{k} = D_{\mu,n}(h)\frac{1+v^2}{(1-v^2)^2} \cdot \frac{1}{s(t)(e_r^2+1)^{\frac{3}{2}}}\zeta(s(t), L, y, y_r), \end{cases} \tag{18}$$

where $S(t) = [s(t), \dot{s}(t), \ldots, s^{(n-1)}(t)]^{\mathrm{T}}$ and $S_{[i]}(t) = [s(t), \dot{s}(t), \ldots, s^{(i-1)}(t)]^{\mathrm{T}}$.

In (18), $\alpha_n(\cdot)$ is generated recursively as follows:

$$\begin{cases} \alpha_1(\cdot) = N(k) \cdot \zeta(s(t), L, y, y_r), \\ \alpha_2(\cdot) = \rho_2\big(S_{[2]}(t), k, L, y, y_r, \hat{\xi}_{[2]}\big), \\ \alpha_i(\cdot) = \rho_i\big(S_{[i]}(t), k, L, y, y_r, \hat{\xi}_{[i-1]}\big) \\ \qquad\quad -z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j, \quad i = 3, \ldots, n, \end{cases} \tag{19}$$

where the design function $\zeta(\cdot)$, which belongs to $\mathcal{C}^n$, is designed as

$$\zeta(\cdot) = \operatorname{sgn}_{\frac{\mu}{2},n}(h)\Big(1 + \bar{\phi}_1(y) + L^2 + M_s\frac{(e_r^2+1)^{\frac{3}{2}}}{s(t)}\Big),$$

with M_s the known bound of $\dot{s}(t)$ as above.

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Intermediate variables z_i's and associated functions $\rho_i(\cdot)$'s are defined as (for $i = 2, \ldots, n$)

$$\begin{cases} z_i \;=\; \hat{\xi}_i - \alpha_{i-1}, \\ \rho_i(\cdot) \;=\; -\frac{1}{2}z_i + L^i a_i \hat{\xi}_1 + \frac{\partial \alpha_{i-1}}{\partial L}\dot{L} + \frac{\partial \alpha_{i-1}}{\partial k}\dot{k} + \sum_{j=0}^{i-2} \frac{\partial \alpha_{i-1}}{\partial s^{(j)}(t)} s^{(j+1)}(t) \\ \qquad\quad -\frac{1}{4}z_i(\frac{\partial \alpha_{i-1}}{\partial y_r})^2 - \frac{1}{4}z_i(\frac{\partial \alpha_{i-1}}{\partial y})^2\big(\hat{\xi}_2^2 + nL^3 + \bar{\phi}_1^2(y)\big). \end{cases} \tag{20}$$

We next intend to present two propositions which show the intuitions behind the design functions/gains above, and which more importantly, serve the performance analysis later on.

Proposition 2. Let $V_z = V_\varepsilon + \frac{1}{2}\sum_{j=2}^n z_i^2$. Then there is

$$\dot{V}_z \;\leqslant\; -\frac{1}{2}L\|\varepsilon\|^2 - \frac{1}{2}\sum_{j=2}^n z_i^2 + \theta_z, \tag{21}$$

with an unknown positive constant θ_z.

Proposition 3. With $D_{\mu,n}(\cdot)$ and $Q_{\frac{\mu}{2},n}(\cdot)$, let V_h be

$$V_h \;=\; \int_0^h D_{\mu,n}(\tau)\,\mathrm{d}\tau \;=\; \begin{cases} \frac{1}{2}(|h|-\mu)^2 + c_\lambda, & |h| \geqslant \frac{3\mu}{2}, \\ \int_{\frac{\mu}{2}}^{|h|} Q_{\frac{\mu}{2},n}(\tau-\mu)\,\mathrm{d}\tau, & \frac{\mu}{2} < |h| < \frac{3\mu}{2}, \\ 0, & |h| \leqslant \frac{\mu}{2}, \end{cases}$$

where $c_\lambda = \int_{\frac{\mu}{2}}^{\frac{3\mu}{2}} Q_{\frac{\mu}{2},n}(\tau-\mu)\,\mathrm{d}\tau - \frac{\mu^2}{8}$. Then there is

$$\dot{V}_h \;\leqslant\; \big(gN(k) + \theta_h\big)\dot{k} + \theta_h \dot{k} V_z^{\frac{1}{2}}. \tag{22}$$

Proof of Proposition 2. Let $V_{z_i} = \frac{1}{2}z_i^2$, $i = 2, \ldots, n$ and take their time derivatives:

$$\dot{V}_{z_i} \;=\; z_i(\dot{\hat{\xi}}_i - \dot{\alpha}_{i-1}) \;=\; z_i\bigg(z_{i+1} + \alpha_i - L^i a_i \hat{\xi}_1 - \frac{\partial \alpha_{i-1}}{\partial k}\dot{k} - \frac{\partial \alpha_{i-1}}{\partial L}\dot{L} - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j \\ - \sum_{j=0}^{i-2}\frac{\partial \alpha_{i-1}}{\partial s^{(j)}(t)} s^{(j+1)}(t)\bigg) - z_i\frac{\partial \alpha_{i-1}}{\partial y}\dot{y} - z_i\frac{\partial \alpha_{i-1}}{\partial y_r}\dot{y}_r. \tag{23}$$

All the terms in the brackets in (23) are available and, hence, can be precisely eliminated by the virtual controller $\alpha_i(\cdot)$ without further estimates. We thus just have to estimate the last two terms in (23). Now, recall that $\dot{y} = \xi_2 + \phi_1(y) = g\hat{\xi}_2 - L^2\varepsilon_2 + \phi_1(y)$. Then, by using $|\phi_1(y)| \leqslant \bar{\theta}\bar{\phi}_1(y)$ and $|\dot{y}_r| \leqslant M_r$, we can learn (using completing the square)

$$\begin{cases} z_i\frac{\partial \alpha_{i-1}}{\partial y}\dot{y} \;=\; z_i\frac{\partial \alpha_{i-1}}{\partial y}\big(g\hat{\xi}_2 - L^2\varepsilon_2 + \phi_1(y)\big) \\ \qquad\qquad\;\leqslant \frac{L}{n}\|\varepsilon\|^2 + \frac{1}{4}z_i^2\big(\hat{\xi}_2^2 + nL^3 + \bar{\phi}_1^2(y)\big)(\frac{\partial \alpha_{i-1}}{\partial y})^2 + g^2 + \bar{\theta}^2, \\ z_i\frac{\partial \alpha_{i-1}}{\partial y_r}\dot{y}_r \;\leqslant\; \frac{1}{4}z_i^2(\frac{\partial \alpha_{i-1}}{\partial y_r})^2 + M_r^2. \end{cases}$$

Putting this and $\alpha_i(\cdot)$ into (23) directly arrives at, for $i = 2, \ldots, n$

$$\dot{V}_{z_i} \;\leqslant\; \frac{L}{n}\|\varepsilon\|^2 - z_{i-1}z_i + z_i z_{i+1} - \frac{1}{2}z_i^2 + g^2 + \bar{\theta}^2 + M_r^2.$$

From this and $\dot{V}_\varepsilon$ in (14), we see (by noting $z_1 = 0$ and $z_{n+1} = 0$) that (21) holds with $\theta_z = (n-1)(g^2 + \bar{\theta}^2 + M_r^2) + \theta_\varepsilon$. □

Proof of Proposition 3. Computing $\dot{V}_h$ and noting the definition of $h(\cdot)$ in (16), we find

$$\dot{V}_h \;=\; D_{\mu,n}(h)\dot{h} \;=\; D_{\mu,n}(h)\frac{1+v^2}{(1-v^2)^2}\cdot\frac{1}{s(t)(e_r^2+1)^{\frac{3}{2}}}\Big(g\alpha_1 + gz_2 - L^2\varepsilon_2 \\ + \phi_1(y) - \dot{y}_r\Big) + D_{\mu,n}(h)\frac{1+v^2}{(1-v^2)^2}\cdot\frac{w(e_r)\dot{s}(t)}{s^2(t)}. \tag{24}$$

Recall from (10) and Proposition 2 that $L(t) \geqslant 1$ and $V_z \geqslant \lambda_{\min}(P)\varepsilon_2^2 + \frac{1}{2}z_2^2$ with $\lambda_{\min}(P)$ the minimum eigenvalue of matrix P. It is then clear that $gz_2 - L^2\varepsilon_2 \leqslant L^2(1+|g|)(|z_2| + |\varepsilon_2|) \leqslant \bar{\theta}_h L^2 V_z^{\frac{1}{2}}$

for $\bar{\theta}_h = (1+|g|)(\sqrt{2} + \frac{1}{\sqrt{\lambda_{\min}(P)}})$. This, together with the sufficient smoothness of $|D_{\mu,n}(h)|$, indicates that the indefinite term "$D_{\mu,n}(h)\frac{1+v^2}{(1-v^2)^2}\cdot\frac{(gz_2-L^2\varepsilon_2)}{s(t)(e_r^2+1)^{\frac{3}{2}}}$" (denoted by ① below) in (24) satisfies

$$① \leqslant |D_{\mu,n}(h)|\frac{1+v^2}{(1-v^2)^2}\cdot\frac{\bar{\theta}_h L^2 V_z^{\frac{1}{2}}}{s(t)(e_r^2+1)^{\frac{3}{2}}}.$$

Now, notice that $|\phi_1(y)| \leqslant \bar{\theta}\bar{\phi}_1(y)$ in (8), $|\dot{y}_r| \leqslant M_r$ in Assumption 2, $w(e_r) < 1$ in (17) and $|\dot{s}(t)| \leqslant M_s$ in Item ***(iii)*** below (4). It immediately follows that (by taking the absolute values)

$$\begin{cases} D_{\mu,n}(h)\frac{1+v^2}{(1-v^2)^2}\cdot\frac{1}{s(t)(e_r^2+1)^{\frac{3}{2}}}\big(\phi_1(y)-\dot{y}_r\big) \\ \qquad \leqslant \max\{\bar{\theta},\, M_r\}|D_{\mu,n}(h)|\frac{1+v^2}{(1-v^2)^2}\cdot\frac{1+\bar{\phi}_1(y)}{s(t)(e_r^2+1)^{\frac{3}{2}}}, \\ D_{\mu,n}(h)\frac{1+v^2}{(1-v^2)^2}\cdot\frac{w(e_r)\dot{s}(t)}{s^2(t)} \leqslant |D_{\mu,n}(h)|\frac{1+v^2}{(1-v^2)^2}\cdot\frac{M_s}{s^2(t)}. \end{cases}$$

Plugging the above estimates, all $\mathcal{C}^n$, and virtual controller $\alpha_1(\cdot)$ (see (19)) into (24) and recalling dynamics $\dot{k}$ in (18) directly arrive at (22) with $\theta_h = \max\{1,\, \bar{\theta},\, M_r,\, \bar{\theta}_h\}$. □

Remark 5. Observe from Lemma 4 that both $D_{\mu,n}(h)$ and $|D_{\mu,n}(h)|$ are of sufficient smoothness, as shown in Figure 2. Benefiting from this key nature, the indefinite terms in (24) can be estimated by simply taking their absolute values. Naturally, this reduces the use of completing the square, which could intuitively result in a less conservative controller.

The concise estimates in the proofs above, compared with the counterpart in [11], should be attributed to the use of dynamic high gain L. In detail, with L at hand, in $\dot{V}_\varepsilon$ we are able to cancel all unwanted highly nonlinear terms, instead of leaving them to the virtual controller α_1 in $\dot{V}_h$, as done in [11]. This and the boundedness of the observer error ε realize the "separation" concerning ε and z_i's, and allow us to estimate the quadratic terms "$\frac{1}{2}z_i^2$" in V_z separately.

On the other side, with the estimates on $\dot{V}_z$ and $\dot{V}_h$, these two Lyapunov function candidates are qualified to unfold the anticipated tracking performance and no other candidate is required additionally. Therefore, this not only makes plain the foregoing control design, but saves a lot of effort in analysis (above and below).

6 Stability and Performance Analysis

This section establishes the boundedness of all states of the closed-loop system and unfolds how the prescribed performance is achieved at length.

Theorem 1. For the uncertain nonlinear system (7) under Assumptions 1 and 2, adaptive output-feedback controller (18) guarantees that for any initial data $\xi_0 \in \mathbf{R}^n$, all states of the resulting closed-loop system (i.e., $\xi(t)$, $\hat{\xi}(t)$, $k(t)$, $L(t)$) are well-defined and bounded on $[0, +\infty)$. Furthermore, the tracking error evolves within the prescribed performance boundary $\mathcal{F}(s(t))$. Particularly, for given times $T_{\bar{\lambda}}$ and T_λ $(0 < T_{\bar{\lambda}} < T_\lambda)$, the tracking error decreases to a *moderately low level* $\bar{\lambda}$ at $T_{\bar{\lambda}}$, while at T_λ, it reaches the *anticipated tracking accuracy* $\lambda\,(\lambda<\bar{\lambda})$, i.e., it stays inside the prescribed strip $[-\lambda,\, \lambda]$ after T_λ.

Proof. With the above-designed adaptive controller, from system (7) and dynamic-high-gain observer (9), it can be seen that the vector field of the resulting closed-loop system is continuous in t and locally Lipschitz in $\big(\xi, \hat{\xi}, k(t), L(t)\big)$ in an open neighborhood of the initial data. Thus, the closed-loop system has a unique solution on a small interval $[0,\, t_s)$ (see Theorem 3.1, page 18 of [31]). Let $[0,\, t_f)$ where $0 < t_f < +\infty$ or $t_f = +\infty$ be the maximal existence interval on which the unique solution exists (see Theorem 2.1, page 17 of [31]). When $0 < t_f < +\infty$, it means $\lim_{t\to t_f}\big(\|\xi(t)\|+\|\hat{\xi}(t)\|+k(t)+L(t)\big) = +\infty$. When $t_f = +\infty$, all closed-loop system states are well-defined on $[0,\, +\infty)$.

From (21) we see there exists a constant $C > 0$ such that on $[0,\, t_f)$

$$\dot{V}_z \leqslant -CV_z + \theta_z,$$

which directly shows the boundedness of V_z on $[0,\, t_f)$. Denote one of its upper bounds as V_z^*. Recalling (22) in Proposition 3, we can immediately have

$$\dot{V}_h \leqslant \big(gN(k) + \theta_h + \theta_h V_z^{*\frac{1}{2}}\big)\dot{k}.$$

Then, by applying Lemma 5, we gain the boundedness of $k(t)$, V_h as well as $\int_0^t \big(gN(k(\tau)) + \theta_h + \theta_h V_z^*\big)\dot{k}(\tau)\,\mathrm{d}\tau$, all on $[0,\, t_f)$.

According to the definition of V_h, we see the boundedness of h implies from (16) and (17) that $|v(t, e_r)| = |\frac{w(e_r)}{s(t)}| < 1$. From this and $s(t) > 0$, we see

$$-s(t) < w(e_r) < s(t), \quad \forall t \in [0,\, t_f). \tag{25}$$

Noting the increasing property of $\mathcal{F}(s(t))$ with respect to $s(t)$ and the fact that $\mathcal{F}(w(e_r)) = e_r$, we can obtain from (25)

$$\mathcal{F}(-s(t)) < e_r < \mathcal{F}(s(t)), \quad \forall t \in (0,\, t_f).$$

Since we have shown a small interval $[0,\, t_s)$ exists on which the closed-loop system runs, we see that e_r is bounded on $[0,\, t_s)$. Then, on $[t_s,\, t_f)$, recalling that $\mathcal{F}(s(t)) < +\infty$ and that $e_r(t)$ always evolves within the prescribed boundary $\mathcal{F}(\cdot)$ (indicated by $|e_r| < \mathcal{F}(s(t))$ given above), we have the boundedness of $e_r(t)$ on $[t_s,\, t_f)$. Therefore, we arrive at its boundedness on $[0,\, t_f)$.

By the definition and boundedness of V_z, we know ε, z_i's are all bounded. With e_r proven bounded above, we can see, from bounded y_r in Assumption 2, the boundedness of y, which implies that L defined in (10) is also bounded. By (19) and (20), we readily come by the boundedness of $\hat{\xi}$. As a result, all of the closed-loop signals are bounded and hence $t_f = +\infty$.

Now, we turn to show the prescribed performance. For this purpose, besides keeping e_r evolving within the performance boundary $\mathcal{F}(s(t))$ on $[0,\, +\infty)$, it remains to show: ***i)*** after the prescribed transient stage (i.e., $[0,\, T_{\bar{\lambda}}]$), the tracking error e_r would decrease to an anticipated moderately low level $\bar{\lambda}$; ***ii)*** after the prescribed time T_λ practical tracking performance is achieved.

To see ***i)***, from the decreasing property of $\mathcal{F}(s(t))$ with respect to t, we have $\mathcal{F}(s(t)) < \mathcal{F}(s(T_{\bar{\lambda}})) = \bar{\lambda}$, $\forall t \in [T_{\bar{\lambda}},\, +\infty)$. Because $|e_r| < \mathcal{F}(s(t))$, we see that e_r is squeezed into a low-level strip $[-\bar{\lambda},\, \bar{\lambda}]$ after $T_{\bar{\lambda}}$.

As for ***ii)***, owing to the evolution of e_r within $\mathcal{F}(s(t))$, the fulfillment of the tracking accuracy boils down to the verification of e_r entering into the strip $[-\lambda,\, \lambda]$ at the prescribed time T_λ. To this end, similar to item ***i)***, recall that $s(t) = s(T_\lambda) = \frac{\lambda}{\sqrt{1+\lambda}}$, $\forall t \geqslant T_\lambda$. Putting this into (4), we see $\mathcal{F}(s(t)) = \mathcal{F}(s(T_\lambda)) = \lambda$, $\forall t \in [T_\lambda,\, +\infty)$. Since there is $|e_r(t)| < \mathcal{F}(s(t))$, $\forall t \in [T_\lambda,\, +\infty) \subset [0,\, +\infty)$, we know $|e_r| < \lambda$ at T_λ, and thereafter it will stay therein forever and hence practical tracking performance is achieved. □

Remark 6. Observed from the performance analysis, the tracking error e_r always evolves within the boundary $\mathcal{F}(s(t))$. Note that $\mathcal{F}(s(t))$ is strictly decreasing in t during the transient stage $[0,\, T_{\bar{\lambda}})$ and the low-level ensured stage $[T_{\bar{\lambda}},\, T_\lambda)$ (see Figure 3 above). Thus, the convergence rate of $\mathcal{F}(s(t))$, which can be an exponential rate or others by picking $s(t)$, is viewed as the convergence rate of tracking error e_r. In particular, since $\mathcal{F}(s(t))$ is constructed by piecing together the multiple functions (see Figure 3), the tracking error e_r can have different convergence rates during its different evolution stages.

7 A Simulation Example

Consider the Josephson junction circuit as follows:

$$m_1\ddot{y} + m_2\dot{y} + m_3 \sin y = u, \tag{26}$$

where m_i's are unknown nonzero parameters, and y is the only measurable quantity.

Now, transform system (26) into (1) by letting $x_1 = y$ and $x_2 = m_1 m_2 \dot{y} + m_2^2 y$. Applying the linear transformation (6), we gather unknown control coefficients together and arrive at the 2-order version of system (7) with $\phi_1(y) = -\frac{m_2}{m_2} y$ and $\phi_2(y) = -m_2 m_3 \sin y$. Apparently, growth condition (8) holds with $\bar{\phi}_1(y) = \sqrt{y^2 + 0.025}$ and $\bar{\phi}_2(y) = \sqrt{\sin^2 y + 0.025}$.

Devise the 2-order observer in the form of (9) with $[a_1,\, a_2]^{\mathrm{T}} = [\,3,\ 2\,]^{\mathrm{T}}$. With the $\bar{\phi}_i(y)$'s in hand, we design dynamic high gain L as in (10) with $\delta_1 = \frac{5}{6}$, $\delta_2 = 2$ and correspondingly, $\|\bar{\phi}(\cdot)\|^2 = y^2 + \sin^2(y) +$

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0.05. Let the dynamics of the high gain k as in (18) where $D_{1,2}(h)$ is with $Q_{\frac{1}{2},2}(h) = -\frac{1}{2}h^4 + \frac{3}{4}h^2 + \frac{1}{2}h - \frac{3}{32}$, associated $\mathrm{sgn}_{\frac{1}{2},2}(h)$ is with $P_2(h) = 12h^5 - 10h^3 + \frac{15}{4}h$, and $N(k) = k^2\cos(k)$.

Set $y_r = \frac{4}{5}\sin\frac{t}{2}$, $\bar{\lambda} = 0.3$, $\lambda = 0.2$, $T_{\bar{\lambda}} = 4$ and $T_\lambda = 5$. Then, $s(t)$ as in (5) is with $\overline{Q}_{0.5,2}(-t+4.5) = -\frac{146}{6311}t^4 + \frac{567}{1402}t^3 - \frac{1942}{841}t^2 + \frac{1177}{261}$ and accordingly we pick its upper bound $M_s = 1.5$.

Based on the high gains and design functions above, we can derive the adaptive controller in the form of (18) by following the proposed control scheme.

To show the proposed scheme lends itself well to shaping the tracking error evolution, we provide a contrast simulation that does not consider transient performance guarantee under the same initial conditions. Select the same observer, high gain L and the associated parameters as above. Without taking the transient behavior of e_r into account, the virtual controller $\alpha_1(\cdot)$ becomes $\alpha_1 = N(k)\zeta(\cdot)$ with $\zeta(\cdot) = \mathrm{sgn}_{\frac{1}{2},2}(e_r)\big(1 + \bar{\phi}_1(y) + L^2\big)$, controller $u = \rho_2(\cdot)$ as in (20) but with $\frac{\partial \alpha_1}{\partial s(t)}\dot{s}(t) = 0$, and the high gain k can reduce to $\dot{k} = D_{1,2}(e_r)\zeta(L, y, y_r)$.

Now let $m_1 = -4$, $m_2 = 1$, $m_3 = -4$, and choose the initial conditions $x_1(0) = 0.8$, $x_2(0) = -1$, $\hat{\xi}_1(0) = 3$, $\hat{\xi}_2(0) = 2$, $L(0) = 1$ and $k(0) = 0.1$.

Simulation results with guaranteed performance are depicted in Figure 4, while the contrast simulation results are in Figure 5. Although both schemes can ensure bounded (x, $\hat{\xi}$, L, k) and the ultimate tracking accuracy, only the proposed scheme confines tracking error e_r to evolving within the prescribed boundary $\mathcal{F}(s(t))$ all the time (see Figure 4), which shows the effectiveness of the proposed scheme. Particularly, in Figure 4, e_r decreases to the level $[-0.3, 0.3]$ before the 4th second and enters into the strip $[-0.2, 0.2]$ very quickly, demonstrating achievement of the expected performance.

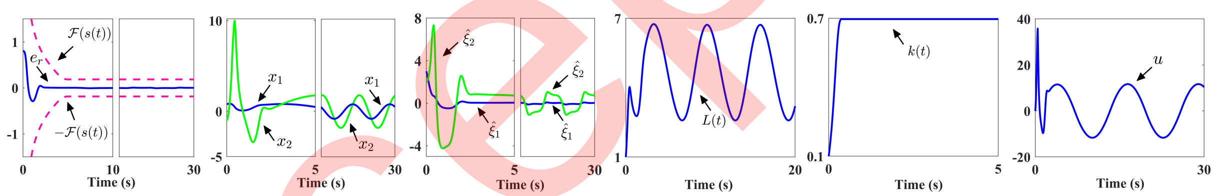

Figure 4 System evolution under proposed guaranteed-performance controller.

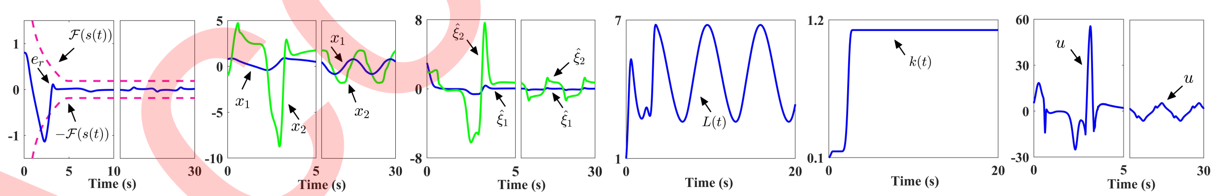

Figure 5 System evolution under the controller without performance guarantee.

8 Concluding Remarks

In this paper, we have addressed *global* output-feedback tracking with arbitrarily prescribed performance for uncertain nonlinear systems. A new adaptive strategy has been put forward where powerful dynamic high gains are leveraged to handle the high nonlinearities and essential uncertainties as well. Since the prescribed performance, characterized in both temporal and spatial aspects, manifests itself in the behavior of the tracking error, a boundary which is built upon spatiotemporal functions has been established to shape it. Notwithstanding, the systems in question are confined to output-feedback-like form. Though we have additionally allowed for unknown control directions, there are more other practical ingredients and implementation realization also worth considering, such as arbitrary relative degree and event-triggered control [2, 24, 26]. Moreover, the systems that can delineate the essence of output feedback in a more generic way, for instance, systems with unmeasured-state-dependent growth [16,19,32], are also intriguing and challenging when considered with prescribed-performance objectives, which deserves further investigation.

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