Received: 10 January 2023 | Revised: 20 July 2023 | Accepted: 17 September 2023

DOI: 10.1002/rnc.7013

RESEARCH ARTICLE

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Continuous adaptive finite-time stabilization with arbitrarily prescribed settling-time

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Funding information
National Natural Science Foundation of China, Grant/Award Numbers: 62033007, 61821004; Taishan Scholars Climbing Program of Shandong Province; Major Fundamental Research Program of Shandong Province

Summary
Rapid convergence that can be prescribed by a user is appealing for many applications with high requirements. This stimulates finite-time stabilization with arbitrarily prescribed settling-time and so-called prescribed-time stabilization. But their continuous realizations had to restrict uncertainties and/or to bear truncated run of controllers. This paper, for nonlinear systems with large uncertainties, realizes not only the convergence within the prescribed time, but also the non-truncated run. First, by lending finite-time stabilization to prescribed-time stabilization and integrating dynamic compensation, an adaptive controller with time-varying components is devised such that the system state reaches the origin at a finite time less than the prescribed time, while exhibiting local asymptotic stability (of the origin). Then by monitoring the finite time online, the time-varying components of the adaptive controller are frozen as their values at the finite time. The asymptotic stability guarantees the frozen adaptive controller can make the system state remain at the origin for all future time. But the above finite time could not be detected in practice, due to ubiquitous disturbances. We thus modified the detection to ensure that the system state enters a vicinity of the origin before the prescribed time and stays there afterwards under some conditions on uncertainties and disturbances. Two simulation examples illustrate the effectiveness of the proposed controller.

KEYWORDS
arbitrarily prescribed settling-time, continuous adaptive control, finite-time stabilization, nonlinear systems, prescribed-time stabilization

1 | INTRODUCTION AND PROBLEM FORMULATION

In this paper, we pursue continuous finite-time stabilization with arbitrarily prescribed settling-time (APST) of the following uncertain nonlinear system:

$$\begin{cases} \dot{\eta}_i = \eta_{i+1} + \varphi_i(t, \eta_{[i]}), & i = 1, \dots, n-1, \\ \dot{\eta}_n = u + \varphi_n(t, \eta), \end{cases} \tag{1}$$

where $\eta = [\eta_1, \dots, \eta_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the system state with the initial condition $\eta_0 = \eta(0)$ and $\eta_{[i]} = [\eta_1, \dots, \eta_i]^{\mathrm{T}}$; $u \in \mathbf{R}$ is the control input; φ_i's are unknown continuous functions satisfying $\varphi_i(t, 0) \equiv 0$, called the system nonlinearities.

We say settling-time is arbitrarily prescribed, which means that for $\forall T_{\rm p} > 0$, the settling-time can be rendered less than $T_{\rm p}$.

Finite-time stabilization, as an appealing control task, is to drive system states to reach zero within a finite time and remain at zero thereafter, specializing in finite-time convergence and robustness and stability arising therefrom. Nevertheless, for global (continuous) finite-time stabilization,[1-8] the settling-time could be rather large due to its heavy dependence on initial conditions, and what's more, it could be unacceptably large when large uncertainties, for which some adaptive technique should be embedded, are allowed in the systems.[6,7] To circumvent the dependence, fixed-time stabilization is subsequently proposed.[9-12] Its settling-time typically owns a uniform bound with respect to initial conditions. Furthermore, by endowing a fixed-time controller with adjustable parameters, it is possible to make its settling-time arbitrarily prescribed (e.g., sufficiently small), and thus to achieve finite-time stabilization with APST. However, the existing continuous design schemes[13,14] precluded large uncertainties as which would render the adjustable parameters inaccessible. As argued in Reference 15, based on fixed-time controllers even strengthened by advanced techniques (e.g., adaptive compensation), it is rather difficult or even impossible, for nonlinear systems with large uncertainties, to achieve global (continuous) finite-time stabilization with arbitrarily prescribed settling-time.

Prescribed-time stabilization was proposed very recently, with the awareness of merits and disabilities of finite-time stabilization.[16-23] It aims to steer system states to zero within arbitrarily prescribed time $T_{\rm p}$, but the involved controllers are only meaningful in the prescribed finite time while literally meaningless beyond the time.[16,17,19-21,23] Prescribed-time stabilization, by making deep use of time-varying gains, paves a disparate route for the convergence within the prescribed time. Its typical ability comes from time-varying gains which could grow to infinity as time tends to the prescribed time $T_{\rm p}$ (i.e., finite-time escape).[16,18,20,23] Moreover, the unbounded time-varying gains can capture any unknown constant in the prescribed time, similar to dynamic high gains.[24,25] This makes possible the prescribed-time controllers (based on the unbounded gains) to accommodate large uncertainties. For instance,[20,21,23] achieved prescribed-time stabilization/regulation for nonlinear systems with large uncertainties by means of suitable time-varying gains. But the controllers do not exhibit good robustness and stability as finite-time controllers.[6,7] Thus, by freezing the time-varying gains as some constants as time sufficiently approaches $T_{\rm p}$, the prescribed-time controllers cannot maintain effective to go beyond $T_{\rm p}$ (for the purpose that the controllers can be extended to the whole horizon). Note that finite-time controllers[6,7] exhibit local asymptotic stability (of the origin), even when they involve dynamic compensation components for large uncertainties. Therefore, different from traditional adaptive controllers, adaptive finite-time controllers could ensure system stability (sufficiently) near the origin of the system state although involved adaptive components (e.g., dynamic high gains) are truncated.

Whether can we achieve continuous adaptive stabilization which integrates the merits of finite-/fixed-time stabilization and prescribed-time stabilization while circumventing their individual demerits, typically for nonlinear systems with large uncertainties? It realizes not only the convergence within the prescribed time, but also the non-truncated run on $[0, +\infty)$ and good robustness and stability as finite-time stabilization.

On this issue, we have already made an attempt via switching adaptive feedback.[15] But the discontinuous feedback achieved therein would confront the matter of implementation. More recently, References 26-31 also realized the non-truncated run, by means of prescribed-time stabilization.[16,17,21] In works,[26,27] the controllers, which tend to zero as time goes to the prescribed finite time, are enforced to keep zero at and after the prescribed time. But in addition to the unexpected unbounded time-varying gains as in prescribed-time stabilization, the zero-constant controllers mean that the systems, after the prescribed time, are in open loop and could not own stability or robustness (e.g., a small noise could cause instability). By contrast, works,[28,29,31] by integrating prescribed-time stabilization and finite-time stabilization, avoided the unbounded time-varying gains and guaranteed stability and robustness during the overall non-truncated run. Specifically, a time-varying (prescribed-time) controller, which is also a finite-time controller before the prescribed finite time, is frozen as its time-invariant counterpart once the states are forced to zero before the prescribed time. Such a line coincides with the line we pursued and later followed. Specially, work[30] borrowed tools of fixed-time stabilization and avoided the unbounded time-varying gains. But in References 28-31, large uncertainties were not involved, and moveover, the nonlinearities were limited to be input matched in References 28-30.

The objective of this paper is, for arbitrarily prescribed time $T_{\rm p} > 0$, to design the following continuous adaptive controller:

$$u = \psi(t, \eta, L, T_{\rm p}), \qquad \dot{L} = \chi(t, \eta, L, T_{\rm p}), \tag{2}$$

where $\psi(\cdot)$ and $\chi(\cdot)$ are scalar continuous functions. The controller (2) shall guarantee that for system (1):

1. *Global boundedness*: all the signals of the resulting closed-loop system are bounded on $[0, +\infty)$ for any initial condition.
2. *Finite-time convergence with APST*: the system state η reaches the origin before the prescribed time and stays there afterwards, namely, for $i = 1, \dots, n$,

$$\lim_{t\to T_{\rm p}} \eta_i(t) = 0, \qquad \eta_i(t) = 0, \quad \forall t \in [T_{\rm p}, +\infty). \tag{3}$$

Despite the elegant objective, this paper still makes the following rather mild assumption on the system nonlinearities.

Assumption 1. There exist known continuous nonnegative functions $\overline{\varphi}_i(\eta_{[i]})$'s with $\overline{\varphi}_i(0) = 0$ such that

$$|\varphi_i(\cdot)| \le \theta\overline{\varphi}_i(\eta_{[i]}) \sum_{j=1}^{i} |\eta_j|^{\frac{r_i-\omega}{r_j}}, \qquad i = 1, \dots, n, \tag{4}$$

where $\theta > 0$ is an unknown constant, $\omega \in \left(0, \frac{1}{n}\right)$ is an even constant* and r_j's are defined by $r_1 = 1$, $r_j = r_{j-1} - \omega, j = 2, \dots, n$.

Assumption 1 indicates that the system nonlinearities accommodate not only large uncertainty "θ", but also low-order growth "$|\eta_j|^{\frac{r_i-\omega}{r_j}}$" for $\frac{r_i-\omega}{r_j} < 1$. Notably, the low-order growth distinguishes the systems from those for prescribed-time stabilization.[16,18,21,23] It can cover linear growth "$|\eta_j|$" in References 21,23, with the presence of the growth rate function $\overline{\varphi}_i(\cdot)$. In fact, the linear growth can be converted into the low-order growth by splitting $|\eta_j| = |\eta_j|^{\frac{r_i-\omega}{r_j}} |\eta_j|^{1-\frac{r_i-\omega}{r_j}}$ and by incorporating $|\eta_j|^{1-\frac{r_i-\omega}{r_j}}$ into the growth rate function. Whereas the low-order growth could not be converted into the linear growth since for the former, its growth rate function, which is just continuous, could not provide power "$1 - \frac{r_i-\omega}{r_j}$" of η_j, for instance $\overline{\varphi}_i(\eta_{[i]}) = \|\eta_{[i]}\|^{\frac{\omega}{2}}$ with $\frac{\omega}{2} < 1 - \frac{r_i-\omega}{r_j}$. Moreover, for the low-order growth itself, there exist some functions that can be bounded by it, such as $|\sin(\eta_j)| \le |\eta_j|^p$ for $p \in (0, 1)$. Thus, due to the coexistence of large uncertainty and low-order growth, system (1) under Assumption 1 could cover various classes of systems (specially practical plants, e.g., simple pendulum system and robotic manipulator[32]).

Although Assumption 1 has appeared in the context of finite-time stabilization,[7] its large uncertainty would definitely not be admitted for APST in global (continuous) finite-time stabilization,[13,14] since as argued above, the large uncertainty would make the adjustable parameters fail to access. On the other hand, the low-order growth in Assumption 1 is ruled out in prescribed-time stabilization.[16-18,21,23] This is because that the low-order growth invalidates the extra exponential convergence[16-18,21] or strict scaling transformation by time-varying gains,[23] either of which is indispensable to achieve a prescribed-time controller. As such, Assumption 1 is said to be rather mild, in comparison with the relevant literature.[13,14,16-18,21,23]

This paper deals with global continuous finite-time stabilization with APST for system (1) under Assumption 1. The expected stabilization cannot be achieved by solely following the scheme in finite-time stabilization on $[0, +\infty)$ or prescribed-time stabilization on $[0, T_{\rm p})$. By integrating the two different stabilizations, we are encouraged to achieve our expected stabilization. On this line,[28-31] are the only works reported, which, however, excluded any large uncertainty. We shall work out a novel strategy, which integrates the ability of the two different stabilizations, to achieve arbitrarily prescribed settling-time (APST) in finite-time stabilization on $[0, +\infty)$ for uncertain systems via continuous adaptive feedback.

The design strategy is split into two steps, respectively deriving two controllers with different actions which work successively. Roughly, in the first step, an adaptive controller with time-varying gains, working on certain $[0, t^*]$ with $t^* < T_{\rm p}$, ensures the system state reaches the origin at t^*, before arbitrarily prescribed time $T_{\rm p}$, while exhibiting asymptotic stability in the vicinity of the origin. In the second step, based on the controller of the first step, we freeze its time-varying gains as their values at t^* to obtain a new controller, which works on $(t^*, +\infty)$ and makes the system state remain at the origin from t^* to infinity.

Specifically, in the first step, we lend finite-time stabilization to prescribed-time stabilization to achieve the aforementioned wanted performances. Most importantly, we work out crucial temporal and state transformations in Section 2. By them, finite t-horizon $[0, T_{\rm p})$ is transformed into infinite τ-horizon $[0, +\infty)$, and correspondingly system (1) is transformed into its certain variant. We devise an adaptive finite-time controller to guarantee that for the variant, the system state reaches the origin in a finite time $\tau^* < +\infty$, while exhibiting asymptotic stability in the vicinity of the origin. With the controller, taking some variable replacements (based on the two transformations), we obtain the adaptive controller with time-varying gains on $[0,\ T_\mathrm{p})$ for system (1). Particularly, the asymptotic stability is retained and $\tau^* < +\infty$ implies the system states reach zero at $t^* < T_\mathrm{p}$ which can be specified by monitoring it online. The first step is truncated at t^*, and the controller for system (1) works on $[0,\ t^*]$.

In the second step starting from t^*, based on the controller of the first step, we freeze its time-varying gains as their values at t^*, thereby obtaining a new adaptive controller on $(t^*, +\infty)$. Owing to the asymptotic stability (of the origin), the new controller can work on $(t^*, +\infty)$ and ensure the zero equilibrium is retained from t^* to infinity. By the control design in two steps, we thus achieve the expected stabilization.

The controller presented above depends critically on the instant t^* which is the first time when the system states exactly reach zero. But in practice, ubiquitous disturbances make that the above finite t^* could not be detected. We thus modified the detection to ensure that the controller, under some conditions on uncertainties and disturbances, can make the system state enter a vicinity of the origin before the prescribed time and stay there afterwards.

2 | TEMPORAL AND STATE TRANSFORMATIONS

In this section, two crucial transformations are introduced. By them, finite-time control with APST is reduced to traditional finite-time control to some extent.

Specifically, the temporal transformation is delineated in Section 2.1 to map finite time interval to infinite time interval, transforming the finite-time convergence into the infinite-time convergence at the expense of certain singularity. Subsequently, a state scaling transformation is introduced in Section 2.2 to reduce the singularity. By the two transformations, the convergence in the prescribed time of system (1) is reduced to the boundedness of system (11), and particularly for system (1), the convergence within the finite time less than the prescribed time can be ensured by the finite-time convergence of system (11).

2.1 | Temporal transformation

Define the following temporal transformation

$$\tau = \mu(t), \tag{5}$$

where $\mu(\cdot)\ :\ \ [0, T_\mathrm{p}) \to [0, +\infty)$ is a twice continuously differentiable function satisfying the following properties:

1. *Strict monotonicity*: The function $\mu(\cdot)$ satisfies $\mu'(t) \geq c_0$ for a known constant $c_0 > 0$, where $\mu'(t) = \frac{\mathrm{d}\mu(t)}{\mathrm{d}t}$.
2. *Ratio boundedness*: The ratio $\frac{\mu''(t)}{\mu'^2(t)}$ satisfies $\frac{|\mu''(t)|}{\mu'^2(t)} \leq c_1$ for a known constant $c_1 > 0$, where $\mu''(t) = \frac{\mathrm{d}^2\mu(t)}{\mathrm{d}t^2}$.

The above transformation originates from the related works.[18,21,23,28] By $\mu_0 = 0$, temporal transformation in our work[23] becomes (5). Particularly, transformation (5), as argued in Reference 23, has different delineation or milder property (***ii***) in comparison with References 18,21, which would give designers more selection freedom. Actually, there are many functions satisfying the above properties. For example, $\mu(t) = \frac{t}{T_\mathrm{p}-t}$ in Reference 21 and $\mu(t) = -T_\mathrm{p} \ln \frac{T_\mathrm{p}-t}{T_\mathrm{p}}$ in Reference 28.

Property (***i***) shows that $\mu(t)$ is invertible. Its inversion is denoted by $\mu^{-1}(\tau)$ for the use.

For later development, define

$$\alpha(\tau) = \mu'(t)\Big|_{t=\mu^{-1}(\tau)}. \tag{6}$$

Note that $\lim_{t\to T_\mathrm{p}} \mu(t) = +\infty$. Then there is $\mu'(t) \to +\infty$ as $t \to T_\mathrm{p}$, otherwise $\mu(t)$ is bounded on $[0, T_\mathrm{p})$. This, together with (5) and (6), implies $\alpha(\tau) \to +\infty$ as $\tau \to +\infty$.

By (6), we have

$$\frac{\mu''(t)}{\mu'(t)} = \frac{\frac{\mathrm{d}\left(\frac{\mathrm{d}\mu(t)}{\mathrm{d}t}\right)}{\mathrm{d}t}}{\frac{\mathrm{d}\mu(t)}{\mathrm{d}t}} = \frac{\mathrm{d}\left(\frac{\mathrm{d}\mu(t)}{\mathrm{d}t}\right)}{\mathrm{d}\mu(t)} = \frac{\mathrm{d}\alpha(\tau)}{\mathrm{d}\tau} =:\ \alpha'(\tau).$$

From properties (***i***) and (***ii***), it follows that

$$\frac{1}{\alpha(\tau)} \leq \frac{1}{c_0}, \quad \left|\frac{\alpha'(\tau)}{\alpha(\tau)}\right| = \frac{|\mu''(t)|}{\mu'^2(t)} \leq c_1. \tag{7}$$

Noting $t = \mu^{-1}(\tau)$, we have $\eta(t) = \eta\left(\mu^{-1}(\tau)\right)$. Furthermore, using the derivation rule of compound function yields

$$\frac{\mathrm{d}\eta\left(\mu^{-1}(\tau)\right)}{\mathrm{d}\tau} = \frac{\mathrm{d}\eta\left(\mu^{-1}(\tau)\right)}{\mathrm{d}\mu^{-1}(\tau)} \cdot \frac{\mathrm{d}\mu^{-1}(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{8}$$

From (5) to (6), it follows that $\mathrm{d}\tau = \frac{\mathrm{d}\mu(t)}{\mathrm{d}t}\mathrm{d}t = \alpha(\tau)\mathrm{d}t$. Then by (8), we can reexpress system (1) in infinite τ-horizon

$$\begin{cases} \frac{\mathrm{d}\eta_i\left(\mu^{-1}(\tau)\right)}{\mathrm{d}\tau} = \frac{1}{\alpha(\tau)}\eta_{i+1} + \frac{1}{\alpha(\tau)}\varphi_i\left(\mu^{-1}(\tau), \eta_{[i]}\right), & i = 1, \ldots, n-1, \\ \frac{\mathrm{d}\eta_n\left(\mu^{-1}(\tau)\right)}{\mathrm{d}\tau} = \frac{1}{\alpha(\tau)}u + \frac{1}{\alpha(\tau)}\varphi_n\left(\mu^{-1}(\tau), \eta\right). \end{cases} \tag{9}$$

By dint of transformation (5), the objective $\lim_{t\to T_p} \eta_i(t) = 0$ in (3) is reduced to the infinite-time convergence of system (9) on $[0, +\infty)$. This enables traditional theories and methods on infinite time interval to be used, and thus renders the convergence within the prescribed time possible for uncertain nonlinear systems.

Nevertheless, transformation (5) makes system (9) be of certain singularity; that is, the control gain confronts factor "$\frac{1}{\alpha(\tau)}$" which tends to zero as $\tau \to +\infty$. This is disadvantageous to the controller design. Thus, it is required to introduce a state scaling transformation to reduce the singularity.[23,33]

2.2 | State scaling transformation

To reduce the singularity caused by the temporal transformation, we introduce state scaling transformation as follows:

$$x_i = \alpha^{n-i+1}(\tau)\eta_i, \qquad i = 1, \ldots, n. \tag{10}$$

By this and (9), we get the following variant of system (1):

$$\begin{cases} \frac{\mathrm{d}x_i(\tau)}{\mathrm{d}\tau} = x_{i+1} + f_i(\tau, x_{[i]}), & i = 1, \ldots, n-1, \\ \frac{\mathrm{d}x_n(\tau)}{\mathrm{d}\tau} = u + f_n(\tau, x), \end{cases} \tag{11}$$

where $f_i(\tau, x_{[i]}) = \alpha^{n-i}(\tau)\varphi_i\left(\mu^{-1}(\tau), \eta_{[i]}\right) + (n-i+1)\frac{\alpha'(\tau)}{\alpha(\tau)}x_i$ with $f_i(\tau, 0) \equiv 0$.

From the definitions of r_i's and $0 < \omega < \frac{1}{n}$ in Assumption 1, it follows that $n - i - (n-j+1)\frac{r_i-\omega}{r_j} = -(i-j+1)\cdot\left(1 - \frac{(n-j+1)\omega}{1-(j-1)\omega}\right) < 0$ for $j = 1, \ldots, i$ and $i = 1, \ldots, n$. This, together with the boundedness of $\frac{1}{\alpha(\tau)}$ in (7), implies

$$\max_{j=1,\ldots,i,\ i=1,\ldots,n} \left\{ \alpha^{n-i-\frac{(n-j+1)\cdot(r_i-\omega)}{r_j}}(\tau) \right\} \leq c_2,$$

for a known constant $c_2 > 0$. As a result, from (7), (10) and Assumption 1, it follows that

$$\begin{aligned} |f_i(\tau, x_{[i]})| &\leq \alpha^{n-i}(\tau)\theta\overline{\varphi}_i\left(\frac{x_1}{\alpha^n(\tau)}, \ldots, \frac{x_i}{\alpha^{n-i+1}(\tau)}\right) \cdot \sum_{j=1}^{i}\left(\frac{|x_j|}{\alpha^{n-j+1}(\tau)}\right)^{\frac{r_i-\omega}{r_j}} + (n-i+1)\frac{\alpha'(\tau)}{\alpha(\tau)}|x_i| \\ &\leq \overline{\theta}\overline{f}_i(\tau, x_{[i]})\sum_{j=1}^{i}|x_j|^{\frac{r_i-\omega}{r_j}}, \qquad i = 1, \ldots, n, \end{aligned} \tag{12}$$

where $\overline{\theta} = \max\{\theta, 1\}$ and known nonnegative function $\overline{f}_i(\tau, x_{[i]}) = c_2\overline{\varphi}_i\left(\frac{x_1}{\alpha^n(\tau)}, \ldots, \frac{x_i}{\alpha^{n-i+1}(\tau)}\right) + (n-i+1)c_1|x_i|^{\frac{\omega}{r_i}}$ is continuous with $\overline{f}_i(\tau, 0) \equiv 0$. Due to $\frac{1}{\alpha(\tau)} \leq \frac{1}{c_0}$ in (7), there exists known smooth function $\tilde{f}_i(x_{[i]}) > 0$ such that $\overline{f}_i(\tau, x_{[i]}) \leq \tilde{f}_i(x_{[i]})$.

With two transformations (5) and (10), we learn that if we can realize global finite-time stabilization of system (11), then a controller with time-varying components can be devised such that the system state of system (1) reaches the origin at a finite time less than the prescribed time T_p, while exhibiting asymptotic stability in the vicinity of the origin. By monitoring the finite time online and freezing the time-varying components in the controller as their values at the finite time, the asymptotic stability guarantees the frozen controller maintains effective and the system state remains at the origin for all future time. As such, finite-time stabilization with APST for system (1) can be established on $[0, +\infty)$. In view of this, we next address global finite-time stabilization of system (11).

Remark 1. The similar line has appeared in References 28-30 wherein the system merely has the input matched nonlinearities and admits the uncertainties with known bounds. Without any transformation, by borrowing tools of finite-time stabilization, work[31] also has achieved the non-truncated run and avoided unbounded time-varying gains, but which merely allows for uncertainties with known bounds and nonlinearities with the linear growth. By contrast, system (1) in the current paper accommodates large uncertainty (without any known bound) and low-order growth, reflected by "θ" and "$|\eta_j|^{\frac{r_i-\omega}{r_j}}$" for $\frac{r_i-\omega}{r_j} < 1$, respectively.

3 CONTROLLER DESIGN

This section seeks for an adaptive controller to achieve finite-time stabilization for the variant of system (1) (i.e., system (11)) in infinite τ-horizon. Based on the controller, it is possible to obtain a finite-time controller with APST for system (1) in infinite t-horizon.

We design the following adaptive finite-time controller

$$\begin{cases} u_f(x, \hat{\Theta}) = x_{n+1}^*(x, \hat{\Theta}), \\ \frac{d\hat{\Theta}}{d\tau} = \sum_{j=1}^{n} \gamma_j(\tau, x_{[j]}, \hat{\Theta}) z_j^{2-\omega}, \quad \hat{\Theta}(0) > 0, \end{cases} \tag{13}$$

where $x_{n+1}^*(x, \hat{\Theta})$ and z_j's are recursively defined by

$$\begin{cases} x_{j+1}^*(x_{[j]}, \hat{\Theta}) = -z_j^{r_{j+1}} \phi_j(x_{[j]}, \hat{\Theta}), \\ z_j = x_j^{\frac{1}{r_j}} - \left(x_j^*(x_{[j-1]}, \hat{\Theta})\right)^{\frac{1}{r_j}}, \end{cases} \tag{14}$$

with $x_1^*(\cdot) = x_{[0]} = 0$, nonnegative functions $\phi_j(\cdot)$'s and $\gamma_j(\cdot)$'s are smooth and continuous, respectively, and $\gamma_j(\tau, 0, \hat{\Theta}) \equiv 0$.

In what follows, $\phi_j(\cdot)$'s and $\gamma_j(\cdot)$'s are those generated recursively in Appendix. Their generation is one of important tasks in the paper, which is deferred to the appendix merely for the compactness.

In (13) and (14), $r_1, \ldots, r_n$ and ω are the same as in Assumption 1, and $r_{n+1} = r_n - \omega$.

We then define the Lyapunov function candidate:

$$V(x, \hat{\Theta}) = \sum_{i=1}^{n} W_i(\cdot), \tag{15}$$

where $W_1(x_1) = \frac{1}{2} z_1^2$ and $W_j(x_{[j]}, \hat{\Theta}), j = 2, \ldots, n$ are defined by

$$W_j(\cdot) = \int_{x_j^*(x_{[j-1]}, \hat{\Theta})}^{x_j} \left(s^{\frac{1}{r_j}} - \left(x_j^*(x_{[j-1]}, \hat{\Theta})\right)^{\frac{1}{r_j}} \right)^{2-r_j} ds. \tag{16}$$

For later development, we characterize the basic properties of $W_i(\cdot)$'s by the following proposition (it is similar to proposition 1 of Reference 7, and hence its proof is omitted here).

Proposition 1. *Functions $W_i(\cdot)$, $i = 2, \ldots, n$ are continuously differentiable and satisfy*

$$2^{\frac{(r_i-1)(2-r_i)}{r_i} - 1} r_i |x_i - x_i^*(\cdot)|^{\frac{2}{r_i}} \le W_i(\cdot) \le 2^{1-r_i} z_i^2. \tag{17}$$

Proposition 2. *Function $V(x, \hat{\Theta})$ is continuously differentiable, and for any fixed $\hat{\Theta}$, is positive definite and radially unbounded with respect to x.*

Proof. By Proposition 1, we have the continuous differentiability and the positive definiteness of $V(x, \hat{\Theta})$. It remains to prove its radial unboundedness with respect to x, which actually can be completed by verifying the radial unboundedness with respect to x_i's. From (16), it follows that $W_1(x_1) \to +\infty$ as $|x_1| \to +\infty$. Then, noting $V(x, \hat{\Theta}) \geq W_1(x_1)$ by (15), we have $V(x, \hat{\Theta}) \to +\infty$ as $|x_1| \to +\infty$. Moreover, it can be seen from (14) that $x_i^*(\cdot)$ merely contains variables $x_{[i-1]}$ and $\hat{\Theta}$. Then, with (17), we get $W_i(x_{[i]}, \hat{\Theta}) \to +\infty$ as $|x_i| \to +\infty$. This, together with $V(\cdot) \geq W_i(\cdot)$, implies $V(x, \hat{\Theta}) \to +\infty$ as $|x_i| \to +\infty$. As such, we have the radial unboundedness of $V(x, \hat{\Theta})$ with respect to x_i's and in turn x for fixed $\hat{\Theta}$. ∎

Proposition 3. *Along the closed-loop system consisting of* (11), (13), *and* (14) *with $\phi_j(\cdot)$'s and $\gamma_j(\cdot)$'s as in Appendix, function $V_n(x, \hat{\Theta}) = V(x, \hat{\Theta}) + \frac{1}{2}\tilde{\Theta}^2$ satisfies*

$$\frac{\mathrm{d}V_n}{\mathrm{d}\tau} \leq -\sum_{j=1}^{n} z_j^{2-\omega}, \tag{18}$$

where $\tilde{\Theta} = \Theta - \hat{\Theta}$ with $\Theta > 0$ being an unknown constant depending on unknown $\overline{\theta}$ in (12).

Proof. The proof is completed with the generation of design functions ϕ_i's and γ_i's in Appendix. Therein, parameter Θ is suitably selected to make (A2), (A6), and (A7) hold, which merely depends on unknown $\overline{\theta}$ in the system nonlinearities and acts as an upper bound of some terms with $\overline{\theta}$. ∎

With the above design and propositions, we have the following two lemmas for the variant (11) in infinite τ-horizon, which play a key role in realizing finite-time stabilization with APST for system (1) in infinite t-horizon.

Lemma 1. *Under adaptive controller* (13), *the system signals $x(\tau)$, $\hat{\Theta}(\tau)$ and $u_{\mathrm{f}}(\tau)$ are globally bounded on* $[0, +\infty)$, *namely, for any initial condition* $(x(0), \hat{\Theta}(0))$, *there is constant $M > 0$ such that* $\|x(\tau)\| + |\hat{\Theta}(\tau)| + |u_{\mathrm{f}}(\tau)| \leq M, \quad \forall \tau \geq 0$.

Proof. From (11), (13), and (14), we see that the vector field of the closed-loop system is continuous in $(\tau, x, \hat{\Theta})$, but not locally Lipschitz in x. Then the closed-loop system has at least one forward solution satisfying the initial condition.

By the radial unboundedness of $V(x, \hat{\Theta})$ in Proposition 2, we have $V_n(x, \hat{\Theta})$ is radially unbounded with respect to $(x, \hat{\Theta})$.

From (18), it follows that

$$0 \leq V_n(x(\tau), \hat{\Theta}(\tau)) \leq V_n(x(0), \hat{\Theta}(0)) < +\infty, \quad \tau \geq 0. \tag{19}$$

Note that $V_n(x(0), \hat{\Theta}(0))$ merely depends on the initial condition. Then, from (19) and the radial unboundedness of $V_n(x, \hat{\Theta})$, it is concluded that all the solutions (starting from the identical initial condition) share one upper bound on $[0, +\infty)$. Thus, all the closed-loop signals are globally bounded on $[0, +\infty)$. ∎

Lemma 2. *Under adaptive controller* (13), *system states x_i's reach zero in finite time and remain at zero thereafter*.

Proof. By (13), (14) and the boundedness of x_i's and $\hat{\Theta}$, we readily get the boundedness of $\frac{\mathrm{d}z_i}{\mathrm{d}\tau}$, which implies the uniform continuity of z_i's. Moreover, using (18) yields $\int_0^{+\infty} z_i^{2-\omega}(\tau)\mathrm{d}\tau < +\infty$ with $2 - \omega > 1$. Thus, by Barbălat lemma in Reference 32, we obtain $\lim_{\tau\to\infty} z_i(\tau) = 0$. By (14) and the convergence of z_1, we immediately have the convergence of x_1 and $x_2^*(\cdot)$. This, in conjunction with the convergence of z_2 and $x_2 = \left(z_2 + \left(x_2^*(\cdot)\right)^{\frac{1}{r_2}}\right)^{r_2}$, implies $\lim_{\tau\to\infty} x_2(\tau) = 0$. From (14) and $\lim_{\tau\to\infty} z_2(\tau) = 0$, it follows that the convergence of x_3^*. By this and the convergence of z_3, and using (14), we get $\lim_{\tau\to\infty} x_3(\tau) = 0$. Similarly, it can be recursively deduced that $\lim_{\tau\to\infty} x_i(\tau) = 0$, $i = 1, \ldots, n$.

Note that function $\gamma_i(\tau, x_{[i]}, \hat{\Theta})$ is continuous and $\gamma_i(\tau, 0, \hat{\Theta}) \equiv 0$. Also note that τ works merely through $\frac{1}{\alpha(\tau)}$ in function $\gamma_i(\cdot)$, and particularly $\frac{1}{\alpha(\tau)} \le \frac{1}{c_0}$. Then, from $\lim_{\tau\to\infty} x_i(\tau) = 0$ and the boundedness of $\tilde{\Theta}$, it is concluded that for any initial condition, there exists a finite time $T_1 > 0$ such that

$$\frac{1}{2} - \tilde{\Theta}\gamma_i(\tau, x_{[i]}(\tau), \hat{\Theta}(\tau)) \ge 0, \quad \tau \ge T_1, \qquad i = 1, \dots, n. \tag{20}$$

By this, (13) and (18), and recalling $V_n(x, \hat{\Theta}) = V(x, \hat{\Theta}) + \frac{1}{2}\tilde{\Theta}^2$, we get (on $[T_1, +\infty)$)

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} \le -\frac{1}{2}\sum_{j=1}^{n} z_j^{2-\omega} - \sum_{j=1}^{n}\left(\frac{1}{2} - \tilde{\Theta}\gamma_j(\cdot)\right)z_j^{2-\omega} \le -\frac{1}{2}\sum_{j=1}^{n} z_j^{2-\omega}. \tag{21}$$

From Proposition 1 and $(|\xi_1| + \cdots + |\xi_n|)^p \le |\xi_1|^p + \cdots + |\xi_n|^p$ for $0 < p \le 1$ and $\xi_i \in \mathbf{R}$, it follows that

$$-\sum_{j=1}^{n} z_j^{2-\omega} \le -2^{\frac{(2-\omega)(r_n-1)}{2}} \sum_{j=1}^{n} W_j^{\frac{2-\omega}{2}}(x_{[j]}, \hat{\Theta}) \le -2^{\frac{(2-\omega)(r_n-1)}{2}} V^{\frac{2-\omega}{2}}(x, \hat{\Theta}). \tag{22}$$

This, together with (21), implies (on $[T_1, +\infty)$)

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} \le -2^{\frac{(2-\omega)(r_n-1)}{2}-1} V^{\frac{2-\omega}{2}}.$$

Thus, $V(x(\tau), \hat{\Theta}(\tau))$ converges to zero in the finite time $T_1 + 2^{1-\frac{(2-\omega)(r_n-1)}{2}} \frac{V^{\frac{\omega}{2}}(x(T_1), \hat{\Theta}(T_1))}{|\omega|} =: T_2$. Note by Proposition 2 that $V(x, \hat{\Theta})$ is positive definite for any fixed $\hat{\Theta}$. Then, the system states $x_i(t)$'s converge to zero in finite time T_2 and remain at zero thereafter. ■

4 | FINITE-TIME STABILIZATION WITH APST

This section presents the main theorem on finite-time stabilization with arbitrarily prescribed settling-time in infinite t-horizon for system (1) under Assumption 1.

Based on (13) and (14), replacing τ and $\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau}$ with $\mu(t)$ and $\frac{1}{\mu'(t)}\dot{\hat{\Theta}}$, respectively, we devise the continuous adaptive controller (t-horizon) as follows:

$$u = \begin{cases} u_{\mathrm{t}}, & \text{if } t \le t^*, \\ u_{\mathrm{c}}, & \text{if } t > t^*, \end{cases} \tag{23}$$

where t^* is generated by

$$t^* = \inf\left\{0 \le t < T_{\mathrm{p}} \;\middle|\; \eta_i(t) = 0,\ i = 1, \dots, n\right\}, \tag{24}$$

which can be specified by monitoring the behaviour of η_i's online. Namely, once the system states reach zero, controller u_{t} is switched into u_{c}.

In (23), u_{t} and u_{c}, based on controller (13), are defined by

$$\begin{cases} \begin{cases} u_{\mathrm{t}} = u_{\mathrm{f}}(x, \hat{\Theta}), \\ \dot{\hat{\Theta}} = \mu'(t)\sum_{j=1}^{n}\gamma_j(\mu(t), x_{[j]}, \hat{\Theta})z_j^{2-\omega}, \quad \hat{\Theta}(0) > 0, \end{cases} \\ \begin{cases} u_{\mathrm{c}} = u_{\mathrm{f}}(x, \hat{\Theta}), \\ \dot{\hat{\Theta}} = \mu'(t^*)\sum_{j=1}^{n}\gamma_j(\mu(t^*), x_{[j]}, \hat{\Theta})z_j^{2-\omega}. \end{cases} \end{cases} \tag{25}$$

In (25), z_j's, $\gamma_j(\cdot)$'s and ω are the same as in (13) and (14), $x = [x_1, \dots, x_n]$, $x_{[j]} = [x_1, \dots, x_j]$, and x_j's are defined by

$$x_j = \begin{cases} \left(\mu'(t)\right)^{n-j+1}\eta_j, & \text{in } u_{\mathrm{t}} \text{ on } t \le t^*, \\ \left(\mu'(t^*)\right)^{n-j+1}\eta_j, & \text{in } u_{\mathrm{c}} \text{ on } t > t^*. \end{cases}$$

Alternatively, controller (23) can be written as

$$\begin{cases} u = u_{\mathrm{f}}(x, \hat{\Theta}), \\ \dot{\hat{\Theta}} = \mu_1(t)\sum_{j=1}^{n}\gamma_j(\mu_2(t), x_{[j]}, \hat{\Theta})z_j^{2-\omega}, \quad \hat{\Theta}(0) > 0, \end{cases}$$

where $x_j = (\mu_1(t))^{n-j+1}\eta_j$,

$$\mu_1(t) = \begin{cases} \mu'(t), & t \le t^*, \\ \mu'(t^*), & t > t^*, \end{cases} \qquad \mu_2(t) = \begin{cases} \mu(t), & t \le t^*, \\ \mu(t^*), & t > t^*, \end{cases}$$

and the others are the same as in (23).

From (25), we see that controller u_{t} builds on not only time-varying gains $\mu(t)$ and $\mu'(t)$ but also adaptive compensation $\hat{\Theta}$. By monitoring online the time t^* at which states η_i's reach zero, we freeze time-varying gains $\mu(t)$ and $\mu'(t)$ as their values at the finite time (i.e., $\mu(t^*)$ and $\mu'(t^*)$), thereby obtaining the adaptive controller u_{c} without time-varying gains. It is worth pointing out that u_{c} seemingly takes value zero at and after the finite time $t^* < T_{\mathrm{p}}$ due to $\eta_i(t) = 0$ for $t \ge t^*$, unlike the related results[26,27] where the controllers are enforced to keep zero at and after the prescribed time.

In particular, controller (23) overcomes the two typical flaws in the literature[16,21,23,26,27] on prescribed-time control: (***i***) The unbounded time-varying gains are avoided, due to that the time-varying gains $\mu(t)$ and $\mu'(t)$ are frozen as constants before the prescribed-time; (***ii***) System (1) under controller (23) always is *of closed loop* on the overall non-truncated run $[0, +\infty)$, and especially controller u_{c} on $(t^*, +\infty)$ is a finite-time controller. Thus, controller (23) can guarantee stability and robustness on $[0, +\infty)$ as finite-time stabilization. But in References 26,27 the zero-constant controllers mean that the systems are *of open loop* after the prescribed time, that is, on $[T_{\mathrm{p}}, +\infty)$. Then the zero-constant controllers, after the prescribed time, could not guarantee stability or robustness (e.g., a small noise could cause instability).

With (23), we are now ready to give the following main theorem.

Theorem 1. *For system* (1) *under Assumption* 1, *the continuous adaptive controller* (23) *guarantees that the system states η_i's converge to zero in arbitrary prescribed time T_{p} and remain at zero thereafter, while all the system signals are bounded on* $[0, +\infty)$.

Before proving Theorem 1, we first give a proposition to show the convergence before the prescribed time T_{p}, whose proof is deferred to behind the proof of Theorem 1.

Proposition 4. *For system* (1) *under Assumption* 1, *the adaptive controller u_{t} defined on* $[0, T_{\mathrm{p}})$ (*in* (25)) *ensures that the system state η converges to the origin in a finite time less than T_{p}, while all the system signals are bounded on* $[0, T_{\mathrm{p}})$.

Proof of Theorem 1. From Proposition 4 and its proof below, we know that under controller u_{t}, the system states η_i's reach zero before $\mu^{-1}(T_2) < T_{\mathrm{p}}$ and remain at zero on $[\mu^{-1}(T_2), T_{\mathrm{p}})$. This implies the existence of t^* and $t^* \le \mu^{-1}(T_2) < T_{\mathrm{p}}$, and thus controller u_{t} is meaningful on $[0, t^*]$.

Note by $\tau = \mu(t)$ that $\dot{V} = \mu'(t)\frac{\mathrm{d}V}{\mathrm{d}\tau}$. Then, from (21) to (22), it follows that on $[0, t^*]$,

$$\dot{V} \le -\frac{\mu'(t)}{2}2^{\frac{(2+\omega)(r_n-1)}{2}}V^{\frac{2-\omega}{2}} - \mu'(t)\sum_{i=1}^{n}\left(\frac{1}{2} - \tilde{\Theta}\gamma_i(\cdot)\right)z_i^{2-\omega}. \tag{26}$$

By (20), we have $\frac{1}{2} - \tilde{\Theta}\gamma_i\left(\mu(t), x_{[i]}(\mu(t)), \hat{\Theta}(\mu(t))\right) > 0$ for $t \in [\mu^{-1}(T_1), t^*]$. This, together with (26) and $\mu'(t) > c_0$, implies (on $[\mu^{-1}(T_1), t^*]$)

$$\dot{V} \le -\frac{c_0}{2}2^{\frac{(2+\omega)(r_n-1)}{2}}V^{\frac{2-\omega}{2}}. \tag{27}$$

From (26), we see that once the system state η enters a enough small neighborhood of the origin to ensure sufficient small $\gamma_i(\cdot)$'s, the system state η remains inside. Namely, the closed-loop system exhibits asymptotic stability of the origin. As such, when one freezes time-varying signals $\mu(t)$ and $\mu'(t)$ in controller u_{t} as constants $\mu(t^*)$ and $\mu'(t^*)$ after t^*; that is, controller u_{t} is transformed into u_{c} after t^*, controller u_{c} works on $(t^*, +\infty)$ and ensures that (27) still holds on $(t^*, +\infty)$. This implies on $[t^*, +\infty)$, zero equilibrium is retained and all the system signals u, $\hat{\Theta}$ and η_i's are bounded. ■

Proof of Proposition 4. Note by (5) that $\tau = \mu(t)$ and $\mathrm{d}\tau = \mu'(t)\mathrm{d}t$. Then, by replacing τ and $\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau}$ with $\mu(t)$ and $\frac{\dot{\hat{\Theta}}}{\mu'(t)}$, controller $u_{\mathrm{f}}(x, \hat{\Theta})$ in (13) defined on $[0, +\infty)$ is changed into $u_{\mathrm{t}}(x, \hat{\Theta})$ in (23) defined on $[0, T_{\mathrm{p}})$.

We next show the boundedness of η, $\hat{\Theta}$ and $u_{\mathrm{t}}(x, \hat{\Theta})$. Note by (7) and (10) that $\alpha(\tau) > c_0$ and $x_i = \alpha^{n-i+1}(\tau)\eta_i$. Also note by Lemma 1 that $x(\tau)$, $\hat{\Theta}(\tau)$ and $u_{\mathrm{f}}(x, \hat{\Theta})$ are bounded on $[0, +\infty)$. Then, we have $\eta(t)$, $\hat{\Theta}(t)$ and $u_{\mathrm{t}}(x, \hat{\Theta})$ are bounded on $[0, T_{\mathrm{p}})$.

It remains to prove the finite-time convergence of η. From Lemma 2, it follows that $x_i\left(\mu^{-1}(\tau)\right)$'s converge to zero before T_2 and remain at zero thereafter. This, together with $x_i = \alpha^{n-i+1}(\tau)\eta_i$ and $\alpha(\tau) > c_0 > 0$, implies that $\eta_i\left(\mu^{-1}(\tau)\right)$'s converge to zero before T_2 and remain at zero on $[T_2, +\infty)$. Note by (5) that $\tau \to T_2$ if and only if $t \to \mu^{-1}(T_2)$ and $\tau \to +\infty$ if and only if $t \to T_{\mathrm{p}}$. Therefore, it can be concluded that $\eta_i(t)$'s converge to zero before $\mu^{-1}(T_2)$ and remain at zero on $\left[\mu^{-1}(T_2), T_{\mathrm{p}}\right)$. ■

5 | ON IMPLEMENTATION OF THE CONTROLLER

For the adaptive controller (23) designed above, it depends critically on the instant t^* which is the first time when the system states exactly reach zero (i.e., $\eta(t^*) = 0$). Actually, in the context of adaptive finite-time control, adaptive controllers exhibit finite-time stability only after the system state enters some small (unknown) neighborhood of the origin.[6,7] And the larger the system uncertainties are, the smaller the neighborhood is. Whereas the unknown θ in Assumption 1 can be arbitrarily large, the neighborhood that is sufficiently small and can be detected is nothing but "zero". This forces us to detect t^* at which the system states are zero instead of sufficiently small.

It is for the interest of theoretical perfection and significance that θ, the system uncertainty of system (1), is assumed to be arbitrarily large and moreover no disturbance is allowed for. However in practice, disturbances are ubiquitous, and system uncertainties (i.e., θ) are relatively small. When disturbances exist, we cannot detect $\eta(t^*) = 0$ as above. Thus we turn to detect $\|\eta(t^*)\| \leq \bar{\varepsilon}$ for a small $\bar{\varepsilon} > 0$, instead of $\eta(t^*) = 0$. We will show that, with such a modified detection, the controller, under some conditions on θ and disturbances, can make the system state enter a neighborhood before the prescribed time and stay there afterwards.

We consider the following input-disturbed version of system (1), that is, unknown $\varphi_n(\cdot)$ therein is replaced by $d(t) + \varphi_n(\cdot)$,

$$\begin{cases} \dot{\eta}_i = \eta_{i+1} + \varphi_i(t, \eta_{[i]}), & i = 1, \ldots, n-1, \\ \dot{\eta}_n = u + d(t) + \varphi_n(t, \eta), \end{cases} \tag{28}$$

where $d(t)$, which is piecewise continuous, represents the input disturbance. The matched disturbance could be due to non-ideal state measurement and control implementation.

Accordingly, growth condition (4) for $i = n$ is replaced by (with constant $\delta \geq 0$)

$$|d(t) + \varphi_n(t, \eta)| \leq \theta \overline{\varphi}_n(\eta) \sum_{j=1}^{n} |\eta_j|^{\frac{r_n - \omega}{r_j}} + \delta. \tag{29}$$

For the input-disturbed system (28) satisfying (29), controller (23), including the detection of t^*, needs to be slightly modified to tolerate additional δ and enable the implementation of the controller.

We should modify (24), that is, the key detection of t^*, as follows

$$t^* = \inf\left\{0 \leq t < T_{\mathrm{p}} \;\middle|\; V(x(t), \hat{\Theta}(t)) \leq \varepsilon\right\}, \tag{30}$$

for a small number $\varepsilon > 0$.

We also modify the dynamics of $\hat{\Theta}$ in controller u_c (defined in (25)) as

$$\dot{\hat{\Theta}}(t) \equiv 0, \qquad t \in (t^*, +\infty), \tag{31}$$

namely, the gain $\hat{\Theta}$ in controller u_c is frozen as constant $\hat{\Theta}(t^*)$.

Theorem 2. *Consider system* (28) *under Assumption* 1 *with replacement* (29). *If δ and θ in* (29) *are known and δ is sufficiently small, then controller* (23) *with modifications* (30) *and* (31) *can ensure that all the system signals are bounded on* $[0, +\infty)$, *and furthermore,* $V(x(t), \hat{\Theta}(t)) \le \varepsilon$, $t \ge T_p$ *for arbitrary prescribed time* $T_p > 0$ *and small number* $\varepsilon > 0$.

Proof. We first show the existence of $t^* < T_p$. By transformations (5) and (10), we obtain the following variant of system (28), that is, system (11) in infinite τ-horizon, but $f_n(\cdot)$ therein is replaced by $d\big(\mu^{-1}(\tau)\big) + f_n(\cdot)$,

$$\begin{cases} \frac{\mathrm{d}x_i(\tau)}{\mathrm{d}\tau} = x_{i+1} + f_i(\tau, x_{[i]}), \qquad i = 1, \ldots, n-1, \\ \frac{\mathrm{d}x_n(\tau)}{\mathrm{d}\tau} = u + d\big(\mu^{-1}(\tau)\big) + f_n(\tau, x). \end{cases} \tag{32}$$

As discussed in Section 2, if controller u_f in (13) for system (32) guarantees $V(x(\tau^*), \hat{\Theta}(\tau^*)) \le \varepsilon$ for $\tau^* < +\infty$, then controller u_t for system (28) can ensure $V(x(t^*), \hat{\Theta}(t^*)) \le \varepsilon$, where $\mu(t^*) = \tau^*$. This implies the existence of t^* can be achieved by ensuring that for system (32), there exists a finite time τ^* such that $V(x(\tau^*), \hat{\Theta}(\tau^*)) \le \varepsilon$.

For system (32), $f_i(\cdot)$, $i = 1, \ldots, n-1$, satisfy (12), and $d\big(\mu^{-1}(\tau)\big) + f_n(\cdot)$ satisfies (by (29))

$$\left| d\big(\mu^{-1}(\tau)\big) + f_n(\tau, x) \right| \le \delta + \overline{\theta}\overline{f}_n(\tau, x) \sum_{j=1}^{n} |x_j|^{\frac{r_i - \omega}{r_j}},$$

with known constant $\overline{\theta} = \max\{1, \theta\}$ and the same $\overline{f}_n(\cdot)$ as in (12). Thus (18) is changed into

$$\begin{aligned} \frac{\mathrm{d}V_n}{\mathrm{d}\tau} &\le -\sum_{j=1}^{n} z_j^{2-\omega} + |z_n|^{2-r_n}\delta \\ &\le -\frac{1}{2}\sum_{j=1}^{n} z_j^{2-\omega} + \frac{r_n - \omega}{2-\omega}\left(\frac{2(2-r_n)}{2-\omega}\right)^{\frac{2-r_n}{r_n-\omega}} \delta^{\frac{2-\omega}{r_n-\omega}} \\ &=: -\frac{1}{2}\sum_{j=1}^{n} z_j^{2-\omega} + \theta^*. \end{aligned} \tag{33}$$

Note by Proposition 3 that $V_n(\cdot) = V(x, \hat{\Theta}) + \frac{1}{2}\tilde{\Theta}^2$ with $\tilde{\Theta} = \Theta - \hat{\Theta}$. Then from (33), it follows that

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} \le -\frac{1}{2}\sum_{j=1}^{n} z_j^{2-\omega} + \theta^* + (\Theta - \hat{\Theta})\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau}. \tag{34}$$

From Proposition 5 below and transformation (5), it follows that $V(x(t^*), \hat{\Theta}(t^*)) \le \varepsilon$ with $\mu(t^*) = \tau^*$. This, together with $\tau^* < +\infty$, implies the existence of t^* and $t^* = \mu^{-1}(\tau^*) < T_p$. Thus, time-varying gains $\mu(t)$ and $\mu'(t)$ in u_t are bounded.

By the boundedness of $\hat{\Theta}(\tau)$ and $x_i(\tau)$'s on $[0, \tau^*]$, and recalling transformations (5), (10), and (14), we readily see the boundedness of $\hat{\Theta}(t)$, $x_i(t)$'s, $\eta_i(t)$'s, and $u_f(x, \hat{\Theta})$ on $[0, t^*]$.

It remains to prove that controller u_c ensures for $t > t^*$, $V(x(t), \hat{\Theta}(t)) \le \varepsilon$. Note that controller u_t is changed into u_c by freezing time-varying gains $\mu(t)$ and $\mu'(t)$ as their fixed values at t^* (i.e., $\mu(t^*)$ and $\mu'(t^*)$), and by freezing dynamic gain $\hat{\Theta}(t)$ as constant $\hat{\Theta}(t^*)$. Then, by (34), $\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau} = \sum_{j=1}^{n}\gamma_j(\cdot)z_j^{2-\omega}$ and $\hat{\Theta} \ge 0$, controller u_c renders (for $t > t^*$)

$$\dot{V} \le -\frac{\mu'(t^*)}{2}\sum_{j=1}^{n} z_j^{2-\omega} + \mu'(t^*)\theta^* + \Theta\mu'(t^*)\sum_{j=1}^{n}\gamma_j(\mu(t^*), x_{[j]}, \hat{\Theta}(t^*))z_j^{2-\omega}, \tag{35}$$

where θ^* depending on δ is sufficiently small.

From Proposition 2, we see function $V(x, \hat{\Theta})$ is positive definite with respect to x for any fixed $\hat{\Theta}$. Then, for small number ϵ, $V(x, \hat{\Theta}) \leq \epsilon$ implies $|x_j| \leq \epsilon_1$ for small number $\epsilon_1 > 0$. Moreover, by (17), we have $|x_j - x_j^*| \leq \frac{1}{r_j} 2^{\frac{r_j}{2} - \frac{(r_j-1)(2-r_j)}{2}} W_j^{\frac{r_j}{2}}(\cdot)$. By this, (14) and $V(\cdot) = \sum_{j=1}^n W_j(\cdot)$, we get $|z_j| \leq \epsilon_2$ for small number $\epsilon_2 > 0$ when $V(\cdot) \leq \epsilon$. Note that $\gamma_j(\tau, 0, \hat{\Theta}) \equiv 0$. Then when x_j's and z_j's are sufficiently small and θ^* is relatively smaller, the first negative term of (35) can dominate the last two positive terms. This implies $\dot{V} \leq 0$ when $V(\cdot) = \epsilon$ for sufficiently small ϵ. Thus $V(x(t), \hat{\Theta}(t)) \leq \epsilon$ for $t > t^*$. ■

Proposition 5. *For system* (32) *in infinite* τ*-horizon, controller* $u_{\rm f}$ *in* (13) *guarantees that there exists a finite time* $\tau^* < +\infty$ *such that* $V(x(\tau^*), \hat{\Theta}(\tau^*)) \leq \epsilon$ *with* ϵ *as in Theorem* 2.

Proof. Suppose by contradiction that no such a time τ^* exists. This means $V(x(\tau), \hat{\Theta}(\tau)) > \epsilon$ on the maximal interval of solution existence. Thus, it follows from (22) and (33) that

$$\frac{dV_n}{d\tau} \leq -2^{\frac{(2-\omega)(r_n-1)}{2}-1} V^{\frac{2-\omega}{2}} + \theta^* < -2^{\frac{(2-\omega)(r_n-1)}{2}-1} \epsilon^{\frac{2-\omega}{2}} + \theta^*.$$

Noting that θ^* is sufficiently small, we in turn have $\frac{dV_n}{d\tau} < -\epsilon_3$ for number $\epsilon_3 > 0$ on the maximal interval of solution existence. From this, we see the maximal interval is $[0, +\infty)$ and

$$V_n(x(\tau), \hat{\Theta}(\tau)) - V_n(x(0), \hat{\Theta}(0)) < -\epsilon_3 \tau, \qquad \forall \tau \in [0, +\infty).$$

This means $V_n(x(\tau), \hat{\Theta}(\tau)) < 0$ for some large τ, which is a contradiction. ■

6 SIMULATION EXAMPLES

In this section, we provide two examples to illustrate the effectiveness of the proposed controller. We first consider the model of pendulum with linear damping:

$$J\ddot{q} + B\dot{q} + Mgl \sin q = u, \tag{36}$$

where the relevant definitions can be found in Reference 32.

The first objective is to ensure that q_i's converge to zero before the prescribed time $T_{\rm p}$ and remain at zero thereafter under the condition that B and M are unknown. We thus introduce transformations $\eta_1 = Jq$ and $\eta_2 = J\dot{q}$. Then system (36) is changed into the following form:

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = u - \frac{B}{J}\eta_2 - Mgl \sin\left(\frac{\eta_1}{J}\right), \end{cases} \tag{37}$$

and the objective is transformed into the prescribed-time convergence of η_i's. Clearly, system (37) satisfies Assumption 1 with $\theta = \max\{1, B, M\}$, $\omega = \frac{2}{21}$, $\overline{\varphi}_1 = 0$ and $\overline{\varphi}_2 = \left(\frac{1}{|J|} + \frac{gl}{|J|^{\frac{19}{21}}}\right) \cdot \left(|\eta_2|^{\frac{\omega}{r_2}} + |\eta_1|^{\omega}\right)$ by $\left|\sin\left(\frac{\eta_1}{J}\right)\right| \leq \left|\frac{\eta_1}{J}\right|^{\frac{19}{21}}$.

By the controller design in Section 3 with $\mu(t) = \frac{1}{5(T_{\rm p}-t)}$, we get design functions $\gamma_1(\cdot) = |x_1|^{\frac{2}{21}}$, $\gamma_2(\cdot) = \beta_{21} + \beta_{22}$, $\phi_1(\cdot) = 2 + \sqrt{1 + \hat{\Theta}^2}(1 + x_1^2)^{\frac{1}{21}}$, and $\phi_2(\cdot) = 2.5 + \sqrt{1 + \hat{\Theta}^2}\gamma_2(\cdot) + \beta_{23}(\cdot) + \beta_{24}(\cdot)$ with $\beta_{21} = 0.8\left((|x_2|^{\frac{2}{17}} + |x_1|^{\frac{2}{21}})(1 + \phi_1^{\frac{5}{7}}(\cdot))\right)^{\frac{8}{5}}$, $\beta_{22} = 1.8\left(|x_1|^{\frac{2}{21}}\phi_1^{\frac{21}{17}}(\cdot)\right)^{\frac{40}{21}}$, $\beta_{23} = 1.4\phi_1^{\frac{21}{17}}(\cdot) + 3.5\phi_1^{\frac{680}{441}}(\cdot)$, and $\beta_{24} = 1.8\left(\frac{\hat{\Theta}}{\sqrt{1+\hat{\Theta}^2}} z_1^{\frac{44}{21}}\right)^{\frac{40}{19}} + 1.4\frac{\hat{\Theta}\gamma_2(\cdot)|z_2|}{\sqrt{1+\hat{\Theta}^2}}|z_1|^{\frac{23}{21}}$.

Choose $J = 1$, $B = 5$, $M\,gl = 1$, $\hat{\Theta}(0) = 1$ and $[\eta_1(0), \eta_2(0)]$ being $[-50, 50]$, $[30, 10]$, $[-20, -20]$ and $[5, -15]$. Once the condition $|\eta_i(t)| \leq 0.0000001$ is detected online, that is, t^* is detected, controller $u_{\rm t}$ is switched to controller $u_{\rm c}$. Thus, we

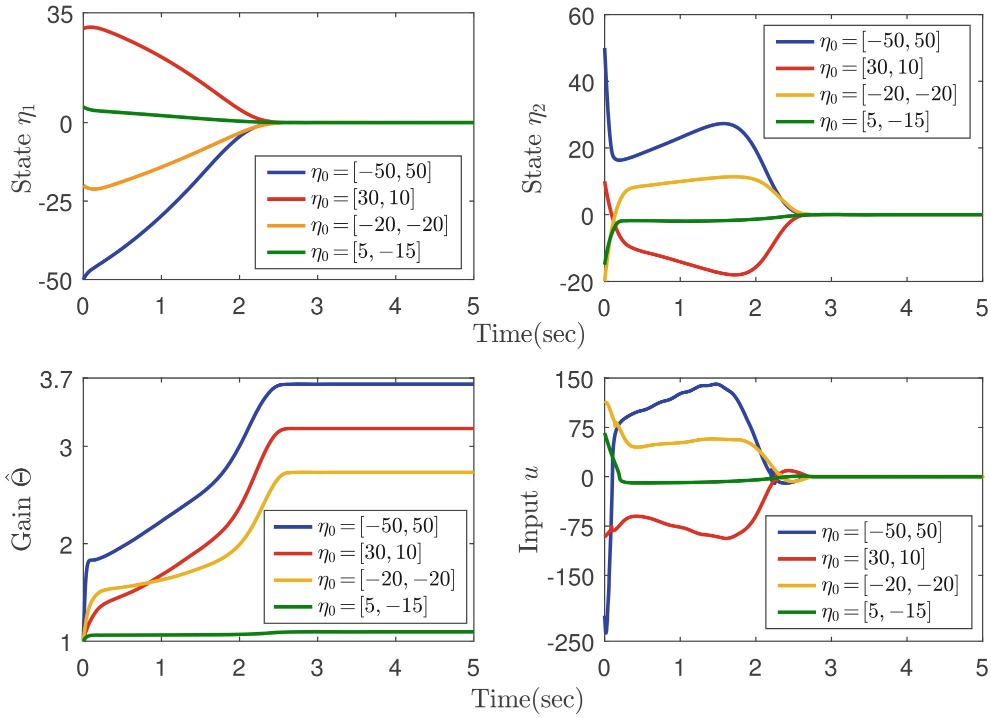

FIGURE 1 Evolution of the closed-loop signals of system (37) with controller (23) for $T_p = 3$ under different initial values.

obtain Figures 1 and 2 for $T_p = 3$ and $T_p = 5$, respectively. Obviously, for every initial value, all the closed-loop signals are bounded, and system states η_i's in Figures 1 and 2 converge to zero before $T_p = 3$ and $T_p = 5$, respectively.

The second objective considers non-vanishing disturbance $\delta \sin(\omega^* t)$ for $\delta > 0$ and $\omega^* > 0$ and measurement noise, due to their ubiquity in practice. We aim to ensure that $(q, \dot{q})$ converges to a vicinity of $(3, 0)$ before the prescribed time T_p and stays there afterwards, under the conditions that constants J, B, M and δ are known and δ is small, and the measurement of η_1 is corrupted with an additive noise of zero mean and standard deviation of 0.001.

Introduce transformations $\eta_1 = J(q-3)$ and $\eta_2 = J\dot{q}$. Then system (36) is changed into the following form:

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = u - \frac{B}{J}\eta_2 - Mgl\sin\left(\frac{\eta_1}{J} + \pi\right) + \delta\sin(\omega^* t), \end{cases} \tag{38}$$

and the objective is transformed into that η_i's converge to a neighborhood of the origin in prescribed time T_p.

Let $J = B = 1, Mgl = 0.1, \delta = 0.001, \omega^* = 3$ and $[\eta_1(0), \eta_2(0), \hat{\Theta}(0)] = [5, -30, 1]$. Choose the detection of t^* as (30) with $\epsilon = 0.1$ and freeze dynamic gain $\hat{\Theta}(t)$ as $\hat{\Theta}(t^*)$ on $(t^*, +\infty)$. Then, by the same design functions as in the first objective, we obtain Figures 3 and 4 for $T_p = 3$ and $T_p = 5$, respectively. It can be seen that all the closed-loop signals are bounded, and particularly, η_i's converge to a vicinity of the origin before the prescribed time.

We next consider the following system typically with low-order growth:

$$\begin{cases} \dot{\eta}_1 = \eta_2 + \theta_1 \eta_1^{\frac{18}{19}}, \\ \dot{\eta}_2 = u + \theta_2 \eta_2^{\frac{16}{17}}, \end{cases} \tag{39}$$

where θ_i's are unknown positive constants. It satisfies Assumption 1 with $\theta = \max\{\theta_1, \theta_2\}$, $\omega = \frac{2}{19}$, $\overline{\varphi}_1(\eta_1) = |\eta_1|^{\frac{1}{19}}$, and $\overline{\varphi}_2(\eta_2) = |\eta_2|^{\frac{1}{17}}$.

For system (39), the control objective is to ensure that η_i's converge to zero before the prescribed time T_p and remain at zero thereafter.

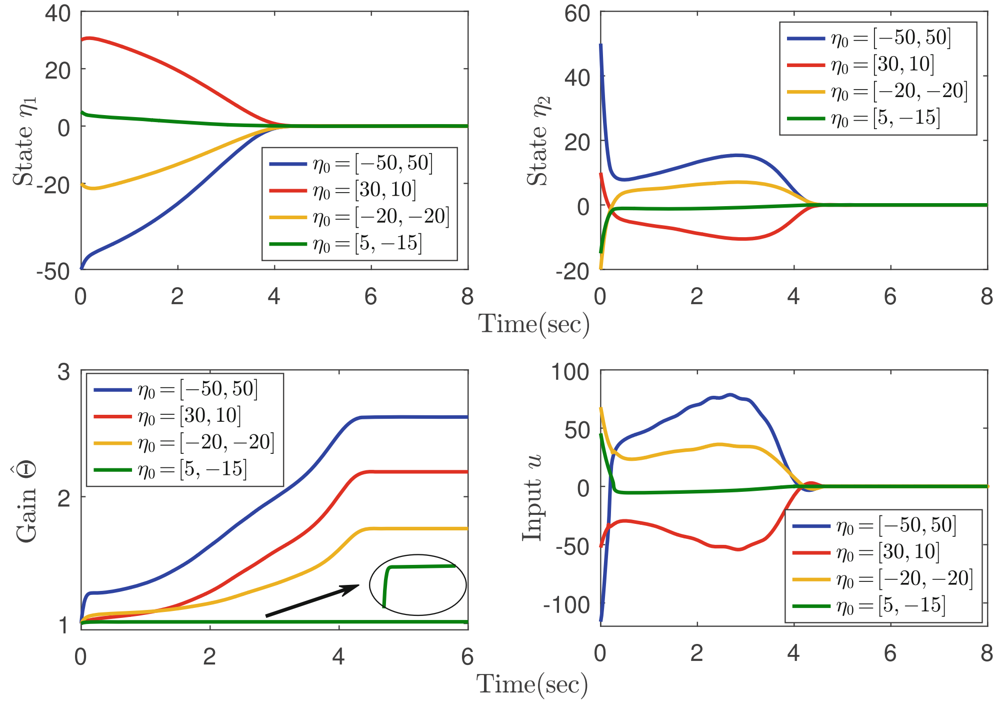

FIGURE 2 Evolution of the closed-loop signals of system (37) with controller (23) for $T_p = 5$ under different initial values.

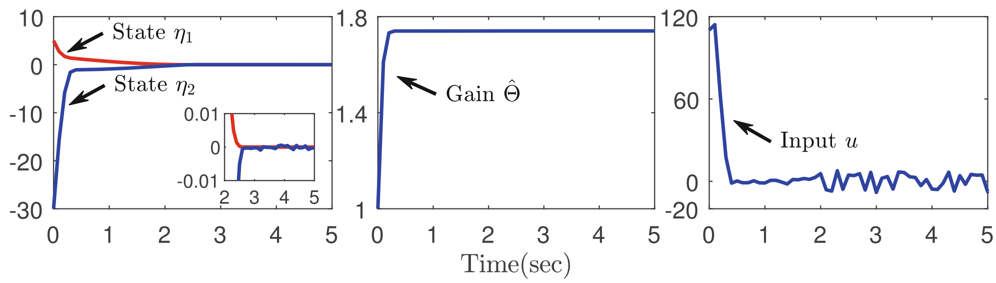

FIGURE 3 Evolution of the closed-loop signals of system (38) under controller (23) with (30) and (31) for $T_p = 3$.

By the controller design in Section 3 with $\mu(t) = \frac{1}{5(T_p - t)}$, we get design functions $\gamma_1(\cdot) = |x_1|^{\frac{1}{19}}$, $\gamma_2(\cdot) = \beta_{21}(\cdot) + \beta_{22}(\cdot)$, $\phi_1(\cdot) = 1 + \sqrt{1+\hat{\Theta}^2}(1+x_1^2)^{\frac{1}{38}}$, and $\phi_2(\cdot) = 2.3 + \hat{\Theta}\gamma_2(\cdot) + \beta_{23}(\cdot) + \beta_{24}(\cdot)$ with $\beta_{21} = 0.8\left((|x_2|^{\frac{2}{17}} + |x_1|^{\frac{4}{19}})(1 + \phi_1^{\frac{15}{19}}(\cdot))\right)^{\frac{36}{23}}$, $\beta_{22} = 1.9\left(|x_1|^{\frac{2}{19}}\phi_1^{\frac{19}{17}}(\cdot)\right)^{\frac{36}{19}}$, $\beta_{23} = 1.1\phi_1^{\frac{19}{17}}(\cdot) + 1.1\phi_1^{\frac{36}{17}}(\cdot)$, and $\beta_{24} = 2.3\left(\frac{\hat{\Theta}}{\sqrt{1+\hat{\Theta}^2}} z_1^2\gamma_1\right)^{\frac{36}{15}} + 1.2\frac{\hat{\Theta}}{\sqrt{1+\hat{\Theta}^2}}\gamma_2|z_1 z_2|$.

We make performance comparisons with the related works.[7,16,21,30] Note that the low-order growth coupling to large uncertainty was not involved in the literature on arbitrarily prescribed settling-time by continuous adaptive feedback. Then the terms "$\theta_1\eta_1^{\frac{18}{19}}$" and "$\theta_2\eta_2^{\frac{16}{17}}$" disable works.[16,21,30] Although for system (39), finite-time controller can be devised,[7] the setting-time may be larger than the prescribed time T_p since the settling-time, depending on the initial condition and system uncertainties, cannot be arbitrarily prescribed. To see this point, we design a finite-time controller by the scheme in Reference 7. Controller therein is $u = -\phi_2(\cdot)z_2^{\frac{15}{19}}$ with $\phi_2(\cdot) = 4.3 + \hat{\Theta}\gamma_2(\cdot) + \beta_{21}(\cdot) + 2.2\phi_1^{\frac{36}{17}}(\cdot)$, $\gamma_2(\cdot) = 0.8|x_2|^{\frac{72}{391}}\phi_1^{\frac{15}{23}}(\cdot) + 1.9|x_1|^{\frac{72}{361}}\phi_1^{\frac{36}{17}}(\cdot)$, $\beta_{21} = 2.3\left(\frac{\hat{\Theta}}{\sqrt{1+\hat{\Theta}^2}} z_1^2\gamma_1\right)^{\frac{36}{15}} + 1.2\frac{\hat{\Theta}}{\sqrt{1+\hat{\Theta}^2}}\gamma_2(\cdot)|z_1 z_2|$, $\dot{\hat{\Theta}} = \gamma_1 z_1^{\frac{36}{19}} + \gamma_2 z_2^{\frac{36}{19}}$, $\gamma_1(\cdot) = |x_1|^{\frac{1}{19}}$, and $\phi_1(\cdot) = 4 + \sqrt{1+\hat{\Theta}^2}(1+x_1^2)^{\frac{1}{38}}$.

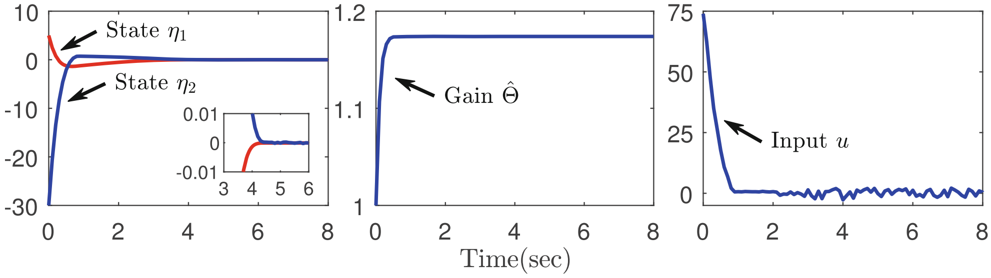

FIGURE 4 Evolution of the closed-loop signals of system (38) under controller (23) with (30) and (31) for $T_p = 5$.

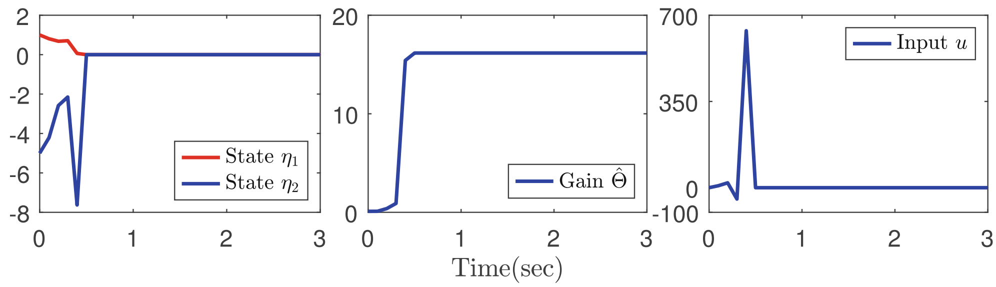

FIGURE 5 Evolution of the closed-loop signals of system (39) with controller (23) for prescribed time $T_p = 0.5$.

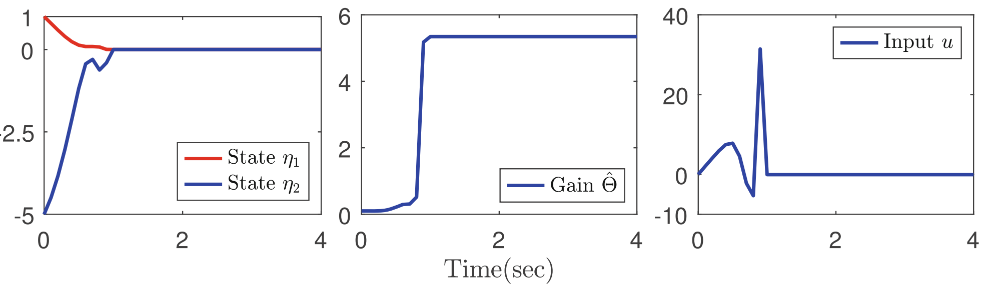

FIGURE 6 Evolution of the closed-loop signals of system (39) with controller (23) for prescribed time $T_p = 1$.

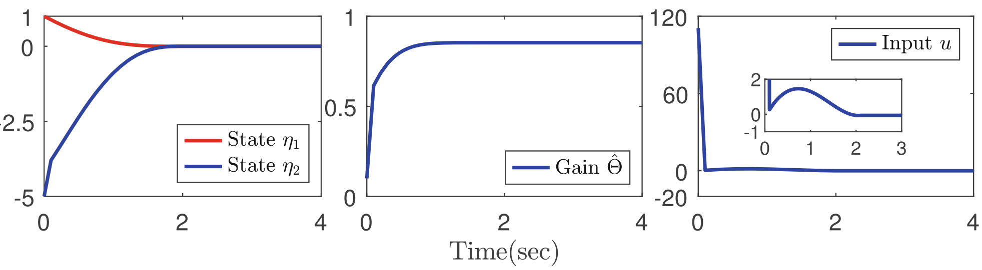

FIGURE 7 Evolution of the closed-loop signals of system (39) with finite-time controller.

Let $\theta_1 = 3$, $\theta_2 = 1$ and $[\eta_1(0), \eta_2(0), \hat{\Theta}(0)] = [1, -5, 0.1]$. Once the condition $|\eta_i(t)| \leq 0.0000001$ is detected online, that is, t^* is detected, controller u_t is switched to controller u_c. Thus, we obtain Figures 5–7, where all the closed-loop signals are bounded. Under the proposed controller of this paper, states η_i's in Figures 5 and 6 converge to zero before $T_p = 0.5$ and $T_p = 1$, respectively. But it can be seen from Figure 7 that finite-time controller only ensures the system states converge to zero in 2 s.

7 | CONCLUDING REMARKS

In this paper, global finite-time stabilization with arbitrarily prescribed setting-time (APST) has been established for uncertain nonlinear systems. Notably, we have introduced elegant temporal and state transformations, reducing the expected stabilization to traditional finite-time stabilization to some extent. By the transformations, an adaptive controller with time-varying components has been devised such that the system states converge to zero within a finite time less than the prescribed time, while exhibiting asymptotic stability (of the origin). Then, by monitoring the finite time online and freezing the time-varying components as their values at the time, we obtain a new adaptive controller to ensure that the system state remains at the origin for all future time. Such a treatment provides a new route for APST, making the systems allow large uncertainty and low-order growth. But the system generality is not comparable to that in the results without time constraints. What extent of the generality can be admitted for finite-time stabilization with APST deserves further investigation. Moreover, only state feedback control has been considered in this paper. While Reference 28 has realized finite-time output feedback stabilization with APST, large uncertainties without any known bound are excluded. Work[20] has involved large uncertainties in the setting of output feedback control, but the controller is defined on the prescribed finite time interval which cannot be extended to infinity. Therefore, it would be of interest to develop a new output feedback scheme, which can ensure APST and the non-truncated run of controllers for general uncertain nonlinear systems.

ACKNOWLEDGMENTS

This work was supported in part by the National Natural Science Foundation of China under grant 62033007 and grant 61821004, in part by the Taishan Scholars Climbing Program of Shandong Province, and in part by the Major Fundamental Research Program of Shandong Province.

CONFLICT OF INTEREST STATEMENT

The authors declare that there is no conflict of interest for this article.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

ENDNOTE

*A number is said to be even if it has even numerator and odd denominator. It is said to be odd if both numerator and denominator are odd.

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How to cite this article: Liu C, Liu Y. Continuous adaptive finite-time stabilization with arbitrarily prescribed settling-time. *Int J Robust Nonlinear Control*. 2023;1-20. doi: 10.1002/rnc.7013

APPENDIX. GENERATION OF DESIGN FUNCTIONS

In this section, we develop a Lyapunov-based recursive procedure to show the generation of design functions ϕ_i's and γ_i's in (13) and (14), and meanwhile verify (18).

For brevity, we introduce notations ($i = 1, \ldots, n$):

$$\Xi_i(\cdot) = \left(\tilde{\Theta} - \sum_{j=2}^{i} \frac{\partial W_j}{\partial \hat{\Theta}}\right) \cdot \left(\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau} - \sum_{j=1}^{i} \gamma_j(\cdot) z_j^{2-\omega}\right). \tag{A1}$$

Step 1. Let $V_1(x_1, \hat{\Theta}) = W_1(x_1, \hat{\Theta}) + \frac{1}{2}\tilde{\Theta}^2$ with $\tilde{\Theta} = \Theta - \hat{\Theta}$. Then, choosing $\Theta \geq \overline{\theta}$, and by (12), we have

$$\frac{\mathrm{d}V_1}{\mathrm{d}\tau} \leq z_1 x_2 + \Theta \overline{f}_1(\tau, x_1) z_1^{2-\omega} - \tilde{\Theta} \frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau}. \tag{A2}$$

Note by (12) that $\tilde{f}_1(x_1) \geq \bar{f}_1(\cdot)$ is a smooth nonnegative function. Then, choosing $\phi_1(\cdot) = n + \sqrt{1+\hat{\Theta}^2}\tilde{f}_1(\cdot)$ and $\gamma_1(\cdot) = \bar{f}_1(\cdot)$, and using (A1) and (A2) yield

$$\frac{\mathrm{d}V_1}{\mathrm{d}\tau} \leq z_1(x_2 - x_2^*) - nz_1^{2-\omega} - \Xi_1(\cdot).$$

Recursive design step $\boldsymbol{i}$ ($\boldsymbol{i = 2, \ldots, n}$). Suppose that steps $1, \ldots, i-1$ have been completed; that is, design functions $\phi_i(\cdot)$'s and $\gamma_i(\cdot)$'s have been found such that $V_{i-1}(x_{[i-1]}, \hat{\Theta}) = V_1(x_1, \hat{\Theta}) + \sum_{j=2}^{i-1} W_j(x_{[j]}, \hat{\Theta})$ satisfies

$$\frac{\mathrm{d}V_{i-1}}{\mathrm{d}\tau} \leq z_{i-1}^{2-r_{i-1}}(x_i - x_i^*) - (n-i+2)\sum_{j=1}^{i-1} z_j^{2-\omega} - \Xi_{i-1}(\cdot). \tag{A3}$$

Let $V_i = V_{i-1} + W_i$. Then, from (A3), it follows that

$$\frac{\mathrm{d}V_i}{\mathrm{d}\tau} \leq z_{i-1}^{2-r_{i-1}}(x_i - x_i^*) - (n-i+2)\sum_{j=1}^{i-1} z_j^{2-\omega} - \Xi_{i-1}(\cdot) + \frac{\partial W_i}{\partial \hat{\Theta}}\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau} + z_i^{2-r_i}x_{i+1} + z_i^{2-r_i}f_i(\cdot) + \sum_{j=1}^{i-1}\frac{\partial W_i}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}\tau}. \tag{A4}$$

We next estimate the first term and the last two terms on the right-hand side of (A4).

Using $|x^p - y^p| \leq 2^{1-p}|x-y|^p$ for odd number $0 < p < 1$ and Young's inequality, by (14) and $r_i = r_{i-1} - \omega$, we have

$$z_{i-1}^{2-r_{i-1}}(x_i - x_i^*) \leq 2^{1-r_i}|z_{i-1}|^{2-r_{i-1}}|z_i|^{r_i} \leq \frac{1}{4}z_{i-1}^{2-\omega} + c_{i1}z_i^{2-\omega}, \tag{A5}$$

where c_{i1} is a positive constant.

Note that $|x+y|^p \leq |x|^p + |y|^p$ for $0 < p < 1$. Then, from (12), (14) and Young's inequality, it follows that

$$\begin{aligned} z_i^{2-r_i}f_i(\cdot) &\leq |z_i|^{2-r_i}\overline{\theta f}_i(\tau, x_{[i]})\sum_{j=1}^{i}|x_j|^{\frac{r_i-\omega}{r_j}} \\ &\leq \overline{\theta f}_i(\cdot)|z_i|^{2-r_i}\sum_{j=1}^{i}|z_j|^{r_i-\omega} + \overline{\theta f}_i(\cdot)|z_i|^{2-r_i}\sum_{j=1}^{i-1}\phi_j^{\frac{r_i-\omega}{r_{j+1}}}(\cdot)|z_j|^{r_i-\omega} \\ &\leq \frac{1}{4}\sum_{j=1}^{i-1}z_j^{2-\omega} + \Theta\beta_{i1}(\tau, x_{[i]}, \hat{\Theta})z_i^{2-\omega}, \end{aligned} \tag{A6}$$

for continuous function $\beta_{i1}(\cdot) \geq 0$ with $\beta_{i1}(\tau, 0, \hat{\Theta}) \equiv 0$.

After some tedious calculations, we have

$$\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}\tau} \leq \frac{1}{4}\sum_{j=1}^{i-1}z_j^{2-\omega} + \Theta\beta_{i2}(\tau, x_{[i]}, \hat{\Theta})z_i^{2-\omega} + \beta_{i3}(x_{[i]}, \hat{\Theta})z_i^{2-\omega}, \tag{A7}$$

where $\beta_{i2}(\cdot)$ is a continuous nonnegative function with $\beta_{i2}(\tau, 0, \hat{\Theta}) \equiv 0$, and $\beta_{i3}(\cdot)$ is a smooth nonnegative function.

Let us next verify (A7). From (14), we obtain

$$\left(x_i^*(x_{[i-1]}, \hat{\Theta})\right)^{\frac{1}{r_i}} = -\sum_{l=1}^{i-1}\prod_{m=l}^{i-1}\phi_m^{\frac{1}{r_{m+1}}}(x_{[m]}, \hat{\Theta})x_l^{\frac{1}{r_l}},$$

which implies

$$\frac{\partial\left(\left(x_i^*(x_{[i-1]}, \hat{\Theta})\right)^{\frac{1}{r_i}}\right)}{\partial x_j} = -\sum_{l=1}^{i-1}\frac{\partial\left(\prod_{m=l}^{i-1}\phi_m^{\frac{1}{r_{m+1}}}(x_{[m]}, \hat{\Theta})\right)}{\partial x_j}x_l^{\frac{1}{r_l}} - \prod_{m=j}^{i-1}\phi_m^{\frac{1}{r_{m+1}}}(\cdot)\frac{1}{r_j}x_j^{\frac{1}{r_j}-1}.$$

Moreover, from (14) and $|x+y|^p \le |x|^p + |y|^p$ for $0 < p < 1$, it follows that

$$\begin{cases} |x_l|^{\frac{1}{r_l}} \le |z_l| + \phi_{l-1}^{\frac{1}{r_l}}(x_{[l-1]}, \hat{\Theta})|z_{l-1}|, \\ |x_j|^{\frac{1}{r_j}-1} \le |z_j|^{1-r_j} + \phi_{j-1}^{\frac{1-r_j}{r_j}}(x_{[j-1]}, \hat{\Theta})|z_{j-1}|^{1-r_j}. \end{cases}$$

Thus, there is

$$\frac{\partial\left(\left(x_i^*(x_{[i-1]}, \hat{\Theta})\right)^{\frac{1}{r_i}}\right)}{\partial x_j} \le \sum_{l=1}^{i-1} \overline{\phi}_l(x_{[i-1]}, \hat{\Theta})|z_l|^{1-r_j}, \tag{A8}$$

with smooth nonnegative functions $\overline{\phi}_l(\cdot)$'s.

From (12), (14) and $|x+y|^p \le |x|^p + |y|^p$ for $0 < p < 1$, it can be deduced that

$$\frac{\mathrm{d}x_j}{\mathrm{d}\tau} = x_{j+1} + f_j(\cdot) \le |z_{j+1}|^{r_{j+1}} + \phi_j(\cdot)|z_j|^{r_{j+1}} + \theta\overline{f}_j(\cdot)\left(\sum_{l=1}^{j}|z_l|^{r_j-\omega} + \sum_{l=1}^{j-1}\phi_l^{\frac{r_j-\omega}{r_{l+1}}}(\cdot)|z_l|^{r_j-\omega}\right). \tag{A9}$$

By $|x^p - y^p| \le 2^{1-p}|x-y|^p$ for odd number $0 < p < 1$, we have

$$\int_{x_i^*(x_{[i-1]},\hat{\Theta})}^{x_i} \left| s^{\frac{1}{r_1}} - \left(x_i^*(x_{[i-1]}, \hat{\Theta})\right)^{\frac{1}{r_i}} \right|^{1-r_i} \mathrm{d}s \le |z_i|^{1-r_i} \cdot \left|x_i - x_i^*(x_{[i-1]}, \hat{\Theta})\right| \le 2^{1-r_i}|z_i|. \tag{A10}$$

Note that

$$\frac{\partial W_i(x_{[i]}, \hat{\Theta})}{\partial x_j} = -\int_{x_i^*(x_{[i-1]},\hat{\Theta})}^{x_i} \left(s^{\frac{1}{r_i}} - \left(x_i^*(x_{[i-1]}, \hat{\Theta})\right)^{\frac{1}{r_i}}\right)^{1-r_i} \mathrm{d}s \cdot (2-r_i)\frac{\partial\left(x_i^*(x_{[i-1]}, \hat{\Theta})\right)^{\frac{1}{r_i}}}{\partial x_j}.$$

Then, by (A8)–(A10) and Young's inequality, we get (A7).

Define $\gamma_i(\cdot) = \beta_{i1}(\cdot) + \beta_{i2}(\cdot)$. Then, by the definition of Ξ_i in (A1), substituting (A5), (A6), and (A7) into (A4) yields

$$\frac{\mathrm{d}V_i}{\mathrm{d}\tau} \le -\left(n - i + \frac{5}{4}\right)\sum_{j=1}^{i-1} z_j^{2-\omega} + z_i^{2-r_i}x_{i+1} - \Xi_i(\cdot) + \left(\hat{\Theta}\gamma_i(\cdot) + c_{i1} + \beta_{i3}(\cdot)\right)z_i^{2-\omega} + \Gamma_i(\tau, x_{[i]}, \hat{\Theta}), \tag{A11}$$

where $\Gamma_i(\cdot) = \sum_{j=2}^{i-1} \frac{\partial W_j}{\partial \hat{\Theta}}\gamma_i(\cdot)z_i^{2-\omega} + \frac{\partial W_i}{\partial \hat{\Theta}}\sum_{l=1}^{i}\gamma_l(\cdot)z_l^{2-\omega}$.

From (A10) and $|x^p - y^p| \le 2^{1-p}|x-y|^p$ for odd number $0 < p < 1$, it follows that

$$\begin{aligned} \left|\frac{\partial W_j(x_{[j]}, \hat{\Theta})}{\partial \hat{\Theta}}\right| &= (2-r_j) \cdot \left|\frac{\partial\left(\left(x_j^*(x_{[j-1]}, \hat{\Theta})\right)^{\frac{1}{r_j}}\right)}{\partial \hat{\Theta}}\right| \cdot \left|\int_{x_j^*(x_{[j-1]},\hat{\Theta})}^{x_j} \left(s^{\frac{1}{r_j}} - \left(x_j^*(x_{[j-1]}, \hat{\Theta})\right)^{\frac{1}{r_j}}\right)^{1-r_j} \mathrm{d}s\right| \\ &\le (2-r_j)2^{1-r_j}\left|\frac{\partial\left(\left(x_j^*(x_{[j-1]}, \hat{\Theta})\right)^{\frac{1}{r_j}}\right)}{\partial \hat{\Theta}}\right| \cdot |z_j|. \end{aligned}$$

This, together with Young's inequality, implies

$$\frac{\partial W_j}{\partial \hat{\Theta}}\gamma_i(\cdot)z_i^{2-\omega} \le (2-r_j)2^{1-r_j}\left|\frac{\partial (x_j^*)^{\frac{1}{r_j}}}{\partial \hat{\Theta}}\right| \cdot |z_j|\gamma_i(\cdot)z_i^{2-\omega} \le \frac{1}{8}z_j^{2-\omega} + \overline{\beta}_{ij}(x_{[i]}, \hat{\Theta})z_i^{2-\omega}, \tag{A12}$$

with smooth function $\overline{\beta}_{ij}(\cdot) \geq 0$. Similarly, we arrive at

$$\frac{\partial W_i}{\partial \hat{\Theta}} \gamma_l(\cdot) z_l^{2-\omega} \leq (2-r_i) 2^{1-r_i} \left| \frac{\partial (x_i^*)^{\frac{1}{r_i}}}{\partial \hat{\Theta}} \right| \cdot |z_i| \gamma_l(\cdot) z_l^{2-\omega} \leq \frac{1}{8} z_l^{2-\omega} + \tilde{\beta}_{il}(x_{[i]}, \hat{\Theta}) z_i^{2-\omega}, \tag{A13}$$

where $\tilde{\beta}_{il}(\cdot)$ is a smooth nonnegative function.

By (A12), (A13) and the definition of $\Gamma_i(\cdot)$ in (A11), we have

$$\Gamma_i(\cdot) \leq \sum_{j=1}^{i-1} \frac{1}{4} z_j^{2-\omega} + \beta_{i4}(x_{[i]}, \hat{\Theta}) z_i^{2-\omega},$$

where $\beta_{i4}(\cdot) = \sum_{j=2}^{i-1} \overline{\beta}_{ij}(\cdot) + \sum_{l=1}^{i} \tilde{\beta}_{il}(\cdot) + \frac{1}{8}$. Substitute this into (A11) and choose $\phi_i(\cdot) = n - i + 1 + \sqrt{1+\hat{\Theta}^2}\,\overline{\gamma}_i(x_{[i]}, \hat{\Theta})$ $+ c_{i1} + \beta_{i3}(\cdot) + \beta_{i4}(\cdot)$ with smooth function $\overline{\gamma}_i(\cdot) \geq \gamma_i(\cdot)$. Then, there is

$$\frac{\mathrm{d}V_i}{\mathrm{d}\tau} \leq -(n-i+1) \sum_{j=1}^{i} z_j^{2-\omega} + z_i^{2-r_i}(x_{i+1} - x_{i+1}^*) - \Xi_i(\cdot). \tag{A14}$$

So far, we have shown how to specify design functions $\phi_i(\cdot)$'s and $\gamma_i(\cdot)$'s in (13) and (14). By (A1) and (A14) with $i = n$, and noting $x_{n+1} = x_{n+1}^*$ and $\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\tau} = \sum_{j=1}^{n} \gamma_j(\cdot) z_j^{2-\omega}$, we readily obtain (18) and Proposition 3.