Prescribed-Time Exact Tracking for a Class of Nonlinear Systems

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***Abstract*—Prescribed time is an appealing but ambitious performance specification for many time-critical applications. Within the prescribed time, *accurately reaching a moving target* naturally becomes a more demanding goal. In this letter, we address *prescribed-time exact tracking* for nonlinear systems. The tracking only works within the prescribed time, differently from the conventional tracking. First, a temporal transformation is performed on the original system, to convert prescribed-time exact tracking in finite horizon to asymptotic tracking in a new infinite horizon. Then, a new vector of augmented reference signals is worked out, with the goal of forcing the difference between such a vector and the state vector of the original system to converge to zero with a desired speed. By scaling the difference to reduce convergence to boundedness, a new weakly time-varying system is obtained which is instrumental for the design of the control law for the original system. In addition, two refined pseudo functions are integrated into Lyapunov functions, thus avoiding the use of completing the square in the design of the control law. The main result is illustrated by a simulation example, after an extension to a class of systems admitting input-matched uncertainties.**

***Index Terms*—Nonlinear systems, prescribed-time exact tracking, temporal and state transformations.**

I. Introduction

TRACKING, owning more practical significance over stabilization in physical world, e.g., automobile cruise and robotic manipulators, has undergone extensive studies during the past few decades [1], [2], [3], [4]. Recently, with the in-depth development of time-critical applications, conventional tracking (practical or asymptotic ones) no longer qualified to be a satisfactory goal in special tracking tasks. For example, an air defense missile should intercept the enemy missile precisely and swiftly, instead of approximately or asymptotically. To fulfill demanding tasks in the applications, tracking targets are required to be reached *precisely* and within a *finite* amount of time, or even more exacting, within an arbitrarily *prescribed* time. Appealing and crucial as the requirements are, they are also ambitious to be met [5], [6].

Notwithstanding, when not confined to tracking or exact tracking, a lot of progress has been made recently in terms of prescribed time for basic goals, e.g., prescribed-time stabilization (PTS) [7], [8], [9], [10], [11], [12], [13], [14], [15] and prescribed-time practical tracking (PTpT) [4], [16]. In view of the unusual specification (i.e., arbitrarily user-prescribed time), the specialized tools are expected to incorporate more powerful enablers. In fact, among the rich results [7], [8], [10], [12], [14], [16], [17], time-varying feedback turns out to be almost necessary for such a time specification, where capable time-varying gains [7], [8], [10], [14], [17] and times-varying performance boundaries [16] are exploited. In detail, works [10], [12], [14], [18] followed a *conversion-based* idea in dealing with PTS — a paramount temporal transformation that stretched the finite horizon to a new infinite one was adopted to convert the PTS into asymptotic stabilization. By doing so, it is possible for many conventional tools to be leveraged [12], [14], [18].

Encouraging as the above results are, things would become quite ambitious when control goals are enhanced to *prescribed-time exact tracking* (PTeT). In comparison with PTS where the target is zero, the target for PTeT no longer stays constant, instead, it is a time-varying signal that changes constantly. This naturally leads to essential time-variants being introduced, regardless of whether the systems are time-varying or not. Moreover, in the PTeT context, states are merely bounded, instead of converging to zero. But the bounded states themselves cannot ensure the inevitable coupling terms (involving unbounded gains) to be bounded and would lead to unbounded control input. This violates the existing PTS [10], [11], [12], [13], [18] (tracking is usually converted to certain stabilization [1], [2], [3]). In comparison with PTpT, PTeT is a completely new type of tracking. It is not enough for the tracking error to enter a pregiven strip within prescribed time, instead, the error needs to hit zero at the prescribed time, a more demanding objective. In addition, once tracking error reaches zero at the prescribed time, the control task of PTeT is completed and the control system ceases to exist [6]. This is different from the conventional tracking in infinite horizon [1], [3], [16]. So far, only two works on PTeT, i.e., [5], [6], have been reported. Work [5] additionally guaranteed non-overshooting response while uncertainties were not considered. Work [6] allowed for uncertainties in MIMO nonlinear systems but the boundedness of control input was actually not guaranteed. Hence, for high applicability, specialized tools are still needed to be developed/refined for PTeT.

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In this letter, we dedicate ourselves to prescribed-time exact tracking for a class of nonlinear systems. Inspired by the conversion-based idea in prescribed-time stabilization, we adopt two transformations — temporal transformation and state temporal scaling. The former, as stated above, shifts the system from the finite horizon to a new infinite horizon, a bridge that connects prescribed-time exact tracking with asymptotic tracking. The latter helps us to further reduce the converted asymptotic tracking to the boundedness of a new scaled system based on which, control design and analysis can be conducted in a concise manner. In the course of control design, a new set of augmented reference signals is worked out. Their difference to system states, i.e., $x_i - \zeta_i$, is forced to converge to zero with a wanted speed. Then, the inevitable coupling terms of the difference with unbounded time-varying gains could be rendered bounded. The new system obtained by scaling the difference, which is weakly time-varying, is instrumental for deriving a capable controller ensuring its bounded states. The use of the augmented signals, in particular, enables the system in question to accommodate essential nonlinearities. Two refined pseudo functions with sufficiently smooth absolute values are integrated into a novel set of Lyapunov functions, by which the use of completing the square in control design is avoided, conceptually leading to a less conservative controller. Moreover, we also extend the nonlinear system to the one with input-matched uncertainties, which shows the capability of our control strategy.

II. Problem Formulation

Arbitrarily prescribing time as users want is an appealing specifications in many applications. Especially, when put in the tracking scenario, exactly tracking the anticipated target within the prescribed time is of crucial and practical interest (even if the system collapses at this time instant).

We confine ourselves to *global exact tracking within prescribed time* for nonlinear systems:

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(x_{[i]}), & i = 1, \dots, n-1, \\ \dot{x}_n = u + f_n(x), \end{cases} \tag{1}$$

where $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the system state vector with initial value $x(0) = x_0$ and $x_{[i]} := [x_1, \dots, x_i]^{\mathrm{T}} \in \mathbf{R}^i$; $u \in \mathbf{R}$ is the control input; $f_i(\cdot)$'s are known smooth nonlinearities.

From the smoothness of $f_i(\cdot)$'s, we can find known smooth nonnegative functions $\bar{f}_i(\cdot)$'s such that

$$\left|f_i(x_{[i]}) - f_i(\zeta_{[i]})\right| \le \bar{f}_i(x_{[i]}, \zeta_{[i]}) \sum_{j=1}^{i} |x_j - \zeta_j|. \tag{2}$$

Assumption 1: Reference signal y_r and its derivatives up to order n, i.e., $\dot{y}_r, \dots, y_r^{(n)}$, are known, continuous and bounded.

In this letter, we are striving to establish global *prescribed-time exact tracking* for system (1). More specifically, the controller is expected to guarantee:

i) the closed-loop systems are well-defined on $[0, T)$ for any arbitrarily prescribed time $T > 0$, and all of the system signals (i.e., state $x(t)$ and control input u) are bounded on $[0, T)$;

ii) tracking error $e_r = x_1 - y_r$ converges exactly to zero within the prescribed time T, that is, $\lim_{t\to T} e_r(t) = 0$.

Note that the time interval of interest is the finite and arbitrarily prescribed $[0, T)$. With the tracking error hitting zero at the prescribed-time T, the input signal and control system cease to work or exist. In fact, for tracking of this type, there are a lot of practical applications. For example, an air defense missile needs to intercept the enemy missile within limited time, and once they collide and explode, they both cease to exist. An aircraft ought to land on the moving carrier by a given time, and once it lands safely, the landing process ceases to exist.

Remark 1: System (1) does not involve any uncertainties. This is for more concise design of prescribed-time exact tracking. In fact, it can admit input-matched uncertainties; see further studies in Section VI.

Remark 2: Let us detail the construction of $\bar{f}_i(\cdot,\cdot)$'s in (2). First, by the mean value theorem, we have $f_i(x_{[i]}) - f_i(\zeta_{[i]}) = f_i'(\nu x_{[i]} + (1-\nu)\zeta_{[i]})(x_{[i]} - \zeta_{[i]})$ for a $\nu \in [0, 1]$. Then, similar to the proof of [19, Lemma 2], we have a known smooth nondecreasing function $\varrho_i\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $|f_i'(\nu x_{[i]} + (1-\nu)\zeta_{[i]})| \le \varrho_i(\|\nu x_{[i]}\|^2 + \|(1-\nu)\zeta_{[i]}\|^2)$. Noting $\nu \in [0, 1]$, we see $\varrho_i(\|x_{[i]}\|^2 + \|\zeta_{[i]}\|^2)$ is one choice of $\bar{f}_i(x_{[i]}, \zeta_{[i]})$ in (2).

III. System Temporal Transformation

According to the control objective above, the tracking error is expected to shrink to zero within the prescribed time. Such a demanding goal naturally disables many well-developed techniques for asymptotic-time control. In view of this, we propose to follow a *conversion-based* idea, with the aid of two crucial transformations. By doing so, the prescribed-time exact tracking can be converted into asymptotic tracking which is further transformed into the boundedness. More importantly, many existing advanced techniques can be leveraged to solve the wanted prescribed-time exact tracking.

A. Temporal Transformation

Temporal transformation is the key bridge that links the finite horizon to the infinite one, based on which a basic framework has been established for prescribed-time control [10], [11], [17]. By constructing an appropriate temporal transformation, we are able to recast prescribed-time objectives as asymptotic ones in the infinite horizon.

Let the temporal transformation be

$$\tau = \mu(t), \tag{3}$$

where $\mu : [0, T) \to [0, +\infty)$ is twice continuously differentiable and fulfills the following properties:

(i) It is strictly increasing and in particular $\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} \ge 1$.

(ii) It has an arbitrarily prescribed ratio bound, i.e., $\frac{|\mu''(t)|}{\mu'^2(t)} \le \sigma$ for a prescribed constant $\sigma > 0$, where $\mu'(t) = \frac{\mathrm{d}\mu(t)}{\mathrm{d}t}$ and $\mu''(t) = \frac{\mathrm{d}^2\mu(t)}{\mathrm{d}t^2}$.

We then define the vital time-varying gain $\gamma(\tau)$:

$$\gamma(\tau) = \mu'(t)\Big|_{t=\mu^{-1}(\tau)}, \tag{4}$$

where $\mu^{-1}(\tau)$ stands for the inversion of $\mu(t)$.

In light of (4), it is clear that

$$\gamma'(\tau) := \frac{\mathrm{d}\gamma(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}\big(\mu'(t)\big)}{\mathrm{d}\mu(t)} = \frac{\mu''(t)}{\mu'(t)}.$$

Remark 3: For the $\mu(t)$ satisfying properties ***(i)*** and ***(ii)***, there are many such functions [10], [12]. An example is

$\mu(t)=\frac{2\kappa^2 T}{\pi}\tan(\frac{\pi t}{2T})$ with $\kappa\geq\max\{1,\sqrt{\frac{\pi}{2\sigma T}}\}$ and correspondingly $\gamma(\tau)=(\frac{\pi}{2\kappa T})^2\tau^2+\kappa^2$. Another example is $\mu(t)=\frac{\kappa t}{1-\frac{t}{T}}$ with $\kappa\geq\max\{1,\frac{2}{\sigma T}\}$, and its associated $\gamma(\tau)=\kappa(\frac{\tau}{\kappa T}+1)^2$.

From $\mu(t)$ and $\gamma(\tau)$ defined above, it follows that

$$\gamma(\tau)\ \geq\ 1,\quad \frac{|\gamma'(\tau)|}{\gamma(\tau)}\ =\ \frac{|\mu''(t)|}{\mu'^2(t)}\ \leq\ \sigma. \tag{5}$$

By (3) and (4), we see $\mathrm{d}t=\frac{1}{\frac{\mathrm{d}\mu(t)}{\mathrm{d}t}}\mathrm{d}\tau=:\frac{1}{\gamma(\tau)}\mathrm{d}\tau$. Then, we stretch the finite time horizon to the infinite one and shift original system (1) to the following system:

$$\begin{cases}\frac{\mathrm{d}x_i(\mu^{-1}(\tau))}{\mathrm{d}\tau}=\frac{1}{\gamma(\tau)}\big(x_{i+1}+f_i(x_{[i]})\big),\ i=1,\ldots,n-1,\\ \frac{\mathrm{d}x_n(\mu^{-1}(\tau))}{\mathrm{d}\tau}=\frac{1}{\gamma(\tau)}\big(u+f_n(x)\big),\end{cases} \tag{6}$$

which now operates on $[0,+\infty)$ in terms of variable τ. More importantly, note that $\tau\to+\infty$ as $t\to T$; thus, once we make sure of the asymptotic e_r and bounded x_i's of the transformed system (6), the prescribed-time exact tracking of system (1) follows at once.

B. State Temporal Scaling

Observing from (6), one can see control coefficient "$\frac{1}{\gamma(\tau)}$" would decay to zero due to the unboundedness of $\gamma(\tau)$. This strong time variants would significantly weaken the control effects, as $\gamma(\tau)$ approaches infinity.

Another underlying byproduct caused by the use of temporal transformation is that unbounded $\gamma(\tau)$ would couple with system states [10], [11], [12], [13], [18]. However, in the tracking context, states x_i's of (6) are merely rendered bounded, rather than converge to zero with a wanted speed. The bounded states themselves cannot ensure the boundedness of the coupling terms, leading to unbounded control input.

To overcome the significant technical difficulties, we first work out a set of augmented reference signals $\zeta_i(\mu^{-1}(\tau))$'s:

$$\begin{cases}\zeta_1\ =\ y_r,\\ \zeta_i\ =\ \gamma(\tau)\frac{\mathrm{d}\zeta_{i-1}}{\mathrm{d}\tau}-f_{i-1}(\zeta_{[i-1]}),\quad i=2,\ldots,n,\end{cases} \tag{7}$$

to which the system states x_i's are forced to converge. As such, we use the convergent $x_i-\zeta_i$, instead of the bounded x_i's themselves, so that the coupling terms can be bounded.

On the role of ζ_i's, first of all, they are the signals to which the states x_i's converge under some controller in infinite τ-horizon. By this and $\zeta_1=y_r$, the exact tracking in finite t-horizon can be achieved.

On the other side, in the tracking context, high nonlinearities $f_i(x_{[i]})$'s do not vanish at $x_1=y_r$. But with the carefully chosen ζ_i's, $f_i(x_{[i]})$ would converge to bounded $f_i(\zeta_{[i]})$ under a controller. In this sense, we say the ζ_i's are indispensable for the compensation for high nonlinearities.

Remark 4: Note from the smoothness of $f_i(\cdot)$'s and the boundedness of $y_r^{(i)}$ that ζ_i's are bounded and so are their derivatives in t-horizon $\frac{\mathrm{d}\zeta_i}{\mathrm{d}t}$'s. This naturally indicates $\frac{\mathrm{d}\zeta_i}{\mathrm{d}\tau}=\frac{1}{\gamma(\tau)}\frac{\mathrm{d}\zeta_i}{\mathrm{d}t}$ converges to zero.

We then define the following state temporal scaling:

$$\bar{x}_1=\gamma^n(\tau)e_r,\ \ \bar{x}_i=\gamma^{n-i+1}(\tau)\big(x_i-\zeta_i\big),\ i=2,\ldots,n, \tag{8}$$

in order to remove $\frac{1}{\gamma(\tau)}$ in control coefficients of (6) and meanwhile to furnish $x_i-\zeta_i$ with a fast convergence speed.

Then by (6) and (7), we arrive at

$$\begin{cases}\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}\tau}=\gamma^{n-i}(\tau)\big(f_i(x_{[i]})-f_i(\zeta_{[i]})\big)+\bar{x}_{i+1}\\ \qquad +\ (n-i+1)\bar{x}_i\frac{\gamma'(\tau)}{\gamma(\tau)},\quad i=1,\ldots,n-1,\\ \frac{\mathrm{d}\bar{x}_n}{\mathrm{d}\tau}=u+f_n(x_{[n]})+\bar{x}_n\frac{\gamma'(\tau)}{\gamma(\tau)}-\gamma(\tau)\frac{\mathrm{d}\zeta_n}{\mathrm{d}\tau}.\end{cases} \tag{9}$$

Up to now, prescribed-time exact tracking of system (1) is converted to prescribed-time stabilization of $(x-\zeta)$-system which is further transformed to the boundedness of system (9).

We can check, from (2), (5) and (8), the time-varying terms containing $\gamma(\tau)$ in system (9) have time-invariant bounds (see, e.g., (16) below). With weak time variants, system (9) lends itself well to derive a bounded controller that can ensure bounded $\bar{x}_i$'s.

IV. Controller Design

In this section, we aim to devise a state-feedback controller for system (9) in infinite τ-horizon. With the aid of state temporal scaling (8), the controller is merely expected to make sure of the boundedness of all $\bar{x}_i$'s.

Two important functions are introduced to support the following controller design, i.e., the *pseudosign function* $\mathrm{sgn}_{\lambda,n}(\cdot)$ and the *pseudo-dead-zone function* $D_{\lambda,n}(\cdot)$ defined in Lemmas 1 and 2 in Appendix, respectively.

Design the controller as follows:

$$u=\alpha_n\big(\bar{x},\vec{y}_r,\vec{\zeta}_2,\ldots,\vec{\zeta}_n\big), \tag{10}$$

where $\alpha_n(\cdot)$ is generated in a recursive manner:

$$\begin{cases}\alpha_1(\bar{x}_1,\vec{y}_{r[1]})\\ \quad=-\,\mathrm{sgn}_{\frac{\lambda}{2},n}(\xi_1)\Big(\lambda+n\sigma(\bar{x}_1^2+1)^{\frac{1}{2}}\\ \qquad+\bar{f}_1(x_1,\zeta_1)(\bar{x}_1^2+1)^{\frac{1}{2}}\Big)-D_{\lambda,n}(\xi_1),\\ \alpha_i(\bar{x}_{[i]},\vec{y}_{r[i]},\vec{\zeta}_{2[i-1]},\ldots,\vec{\zeta}_{i[1]})\\ \quad=-\,\mathrm{sgn}_{\frac{\lambda}{2},n}(\xi_i)\Big(\lambda+(n-i+1)\sigma(\bar{x}_i^2+1)^{\frac{1}{2}}\\ \qquad+\bar{f}_i(x_{[i]},\zeta_{[i]})\sum_{j=1}^{i}(\bar{x}_j^2+1)^{\frac{1}{2}}+|D_{\lambda,n}(\xi_{i-1})|\\ \qquad+\ \sum_{j=1}^{i-1}\bar{f}_j(\cdot,\cdot)\sum_{k=1}^{j}\big((\frac{\partial\alpha_{i-1}}{\partial\bar{x}_j})^2\bar{x}_k^2+1\big)^{\frac{1}{2}}\Big)\\ \qquad-\ D_{\lambda,n}(\xi_i)+\rho_i(\bar{x}_{[i]},\vec{y}_{r[i]},\vec{\zeta}_{2[i-1]},\ldots,\vec{\zeta}_{(i-1)[1]}),\\ \quad i=2,\ldots,n,\end{cases} \tag{11}$$

with $\bar{f}_i(\cdot,\cdot)$'s given in (2), $\vec{y}_r:=[y_r,\frac{\mathrm{d}y_r}{\mathrm{d}\tau},\frac{\mathrm{d}^2y_r}{\mathrm{d}\tau^2},\ldots,\frac{\mathrm{d}^{n-1}y_r}{\mathrm{d}\tau^{n-1}}]^{\mathrm{T}}$, $\vec{\zeta}_i:=[\zeta_i,\frac{\mathrm{d}\zeta_i}{\mathrm{d}\tau},\ldots,\frac{\mathrm{d}^{n-i}\zeta_i}{\mathrm{d}\tau^{n-i}}]^{\mathrm{T}}$, $\vec{y}_{r[j]}:=[y_r,\frac{\mathrm{d}y_r}{\mathrm{d}\tau},\ldots,\frac{\mathrm{d}^{j-1}y_r}{\mathrm{d}\tau^{j-1}}]^{\mathrm{T}}$ and $\vec{\zeta}_{i[j]}:=[\zeta_i,\frac{\mathrm{d}\zeta_i}{\mathrm{d}\tau},\ldots,\frac{\mathrm{d}^{j-1}\zeta_i}{\mathrm{d}\tau^{j-1}}]^{\mathrm{T}}$.

In (11), intermediate variables ξ_i's and associated design functions ρ_i's are designed as (for $i=2,\ldots,n$)

$$\begin{cases}\xi_1=\bar{x}_1,\quad \xi_i=\bar{x}_i-\alpha_{i-1},\\ \rho_i=\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\bar{x}_j}\big(\bar{x}_{j+1}+(n-j+1)\sigma\bar{x}_j\big)\\ \qquad+\ \sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\vec{\zeta}_{j[i-j]}^{\mathrm{T}}}\vec{\zeta}_{j[2,i-j+1]}+\frac{\partial\alpha_{i-1}}{\partial\vec{y}_{r[i-1]}^{\mathrm{T}}}\vec{y}_{r[2,i]},\end{cases} \tag{12}$$

where $\vec{\zeta}_{i[2,i-j+1]}:=[\frac{\mathrm{d}\zeta_j}{\mathrm{d}\tau},\ldots,\frac{\mathrm{d}^{i-j}\zeta_i}{\mathrm{d}\tau^{i-j}}]^{\mathrm{T}}=\frac{\mathrm{d}}{\mathrm{d}\tau}\vec{\zeta}_{j[i-j]}$ and $\vec{y}_{r[2,i]}:=[\frac{\mathrm{d}y_r}{\mathrm{d}\tau},\ldots,\frac{\mathrm{d}^{i-1}y_r}{\mathrm{d}\tau^{i-1}}]^{\mathrm{T}}=\frac{\mathrm{d}}{\mathrm{d}\tau}\vec{y}_{r[i-1]}$.

Remark 5: In deriving controller (11), we particularly exploit two important functions, i.e., the *pseudosign function* $\mathrm{sgn}_{\lambda,n}(\cdot)$ and the *pseudo-dead-zone function* $D_{\lambda,n}(\cdot)$. Observing from their definitions and sufficient smoothness in

Lemmas 1 and 2, we can readily learn that the absolute value of odd function $D_{\lambda,n}(\cdot)$, i.e., $|D_{\lambda,n}(\cdot)| = \mathrm{sgn}_{\frac{\lambda}{2},n}(\cdot)D_{\lambda,n}(\cdot)$, is also sufficiently smooth. By means of this crucial property, we naturally avoid the use of completing the square in controller derivation, which shall be seen in the later development.

We intend to provide a crucial proposition to give some insight into how the controller (10), virtual controls α_i's in (11) and the associated ρ_i's in (12) are derived. Therein, we also show the role of the controller in handling the nonlinearities and the proposition itself shall facilitate the exposure of the prescribed-time performance shortly.

Proposition 1: Let Lyapunov function candidate $V = \sum_{i=1}^{n} V_{\xi_i}$ with V_{ξ_i} defined as follows

$$V_{\xi_i} = \int_0^{\xi_i} D_{\lambda,n}(s)\mathrm{d}s = \begin{cases} \frac{1}{2}(|\xi_i| - \lambda)^2 + c_\lambda, & |\xi_i| \geq \frac{3\lambda}{2}, \\ \int_{\frac{\lambda}{2}}^{|\xi_i|} Q_{\frac{\lambda}{2},n}(s-\lambda)\mathrm{d}s, & \frac{\lambda}{2} < |\xi_i| < \frac{3\lambda}{2}, \\ 0, & |\xi_i| \leq \frac{\lambda}{2}, \end{cases} \tag{13}$$

where $c_\lambda = \int_{\frac{\lambda}{2}}^{\frac{3\lambda}{2}} Q_{\frac{\lambda}{2},n}(s-\lambda)\mathrm{d}s - \frac{\lambda^2}{8}$ and positive constant λ is a design parameter. Then there is

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} \leq -\sum_{i=1}^{n} D_{\lambda,n}^2(\xi_i). \tag{14}$$

Proof: To verify (14), first we define $V_1 = V_{\xi_1}$ and $V_i = V_{i-1} + V_{\xi_i}$, $i = 2, \ldots, n$, and then estimate their time derivatives in a recursive manner.

Step 1: By the definition of V_{ξ_1}, we have

$$\frac{\mathrm{d}V_{\xi_1}}{\mathrm{d}\tau} = D_{\lambda,n}(\xi_1)\Big(\gamma^{n-1}(\tau)\big(f_1(x_1) - f_1(\zeta_1)\big) + \xi_2 + \alpha_1 + n\bar{x}_1\frac{\gamma'(\tau)}{\gamma(\tau)}\Big). \tag{15}$$

By (2), (5) and (8), we estimate the indefinite terms in (15):

$$\begin{cases} D_{\lambda,n}(\xi_1)\gamma^{n-1}(\tau)\big(f_1(x_1) - f_1(\zeta_1)\big) \\ \qquad \leq |D_{\lambda,n}(\xi_1)|\bar{f}_1(x_1,\zeta_1)(\bar{x}_1^2+1)^{\frac{1}{2}}, \\ D_{\lambda,n}(\xi_1)\xi_2 \leq |D_{\lambda,n}(\xi_1)|\big(|D_{\lambda,n}(\xi_2)| + \lambda\big), \\ D_{\lambda,n}(\xi_1)n\bar{x}_1\frac{\gamma'(\tau)}{\gamma(\tau)} \leq n\sigma|D_{\lambda,n}(\xi_1)|(\bar{x}_1^2+1)^{\frac{1}{2}}. \end{cases} \tag{16}$$

Putting the estimates into (15) and invoking $\alpha_1(\cdot)$ in (11) yield

$$\frac{\mathrm{d}V_{\xi_1}}{\mathrm{d}\tau} \leq -D_{\lambda,n}^2(\xi_1) + |D_{\lambda,n}(\xi_1)D_{\lambda,n}(\xi_2)|.$$

Step i ($i \geq 2$): Suppose the first i steps have been verified and we have obtained

$$\frac{\mathrm{d}V_{i-1}}{\mathrm{d}\tau} \leq -\sum_{j=1}^{i-1} D_{\lambda,n}^2(\xi_j) + |D_{\lambda,n}(\xi_{i-1})D_{\lambda,n}(\xi_i)|. \tag{17}$$

To see whether (17) holds similarly for $\frac{\mathrm{d}V_i}{\mathrm{d}\tau}$, we need to estimate $\frac{\mathrm{d}V_{\xi_i}}{\mathrm{d}\tau}$ owing to $V_i = V_{i-1} + V_{\xi_i}$. By the definition of V_{ξ_i} and the time derivative $\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}\tau}$ in (9), we can obtain

$$\begin{aligned}\frac{\mathrm{d}V_{\xi_i}}{\mathrm{d}\tau} &= D_{\lambda,n}(\xi_i)\Big(\gamma(\tau)^{n-i}\big(f_i(x_{[i]}) - f_i(\zeta_{[i]})\big) + \xi_{i+1} \\ &\quad + (n-i+1)\bar{x}_i\frac{\gamma'(\tau)}{\gamma(\tau)} + \alpha_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\bar{x}_j}\frac{\mathrm{d}\bar{x}_j}{\mathrm{d}\tau} \\ &\quad - \sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\vec{\zeta}_{j[i-j]}^{\mathrm{T}}}\vec{\zeta}_{j[2,i-j+1]} - \frac{\partial\alpha_{i-1}}{\partial\vec{y}_{r[i-1]}^{\mathrm{T}}}\vec{y}_{r[2,i]}\Big).\end{aligned} \tag{18}$$

According to (2), the first three bracketed terms in (18) can be estimated in the analogous flavor to (16), and the 5th bracketed term satisfies

$$\begin{aligned}&-D_{\lambda,n}(\xi_i)\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\bar{x}_j}\frac{\mathrm{d}\bar{x}_j}{\mathrm{d}\tau} \\ &\quad\leq -D_{\lambda,n}(\xi_i)\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\bar{x}_j}\big(\bar{x}_{j+1} + (n-j+1)\sigma\bar{x}_j\big) \\ &\quad + |D_{\lambda,n}(\xi_i)|\sum_{j=2}^{i-1}\bar{f}_j(x_{[j]},\zeta_{[j]})\sum_{k=1}^{j}\big((\frac{\partial\alpha_{i-1}}{\partial\bar{x}_j})^2\bar{x}_k^2 + 1\big)^{\frac{1}{2}}.\end{aligned}$$

Putting the estimates and α_i (in (11)) into $\frac{\mathrm{d}V_i}{\mathrm{d}\tau}$, we see, after collecting the terms, (17) holds with $i-1$ replaced by i. Note that when $i = n$, $D_{\lambda,n}(\xi_{n+1}) = 0$, which gives (14) at once. ■

V. Prescribed-Time Performance Analysis

Note that controller (10) is designed in τ-horizon. Its counterpart in t-horizon can be obtained straightforward by replacing τ with $\mu(t)$.

Theorem 1: For nonlinear system (1) under Assumption 1, controller (10) guarantees that for any initial data $x_0 \in \mathbf{R}^n$, the resulting closed-loop system is well-defined on $[0, T)$, and all of the system signals (i.e., system state $x(t)$ and control input u) are bounded on $[0, T)$. Furthermore, within prescribed time T, the tracking error e_r converges to zero and the rest of the states converge to the bounded augmented reference signals ζ_i's; that is, $\lim_{t\to T}(e_r(t), x_2(t), \ldots, x_n(t)) = (0, \zeta_2(T), \ldots, \zeta_n(T))$.

Proof: As explicated above, to show the controller realizes prescribed-time exact tracking and the associated convergence for system (1), it suffices to show controller (10) makes sure system (9) in infinite τ-horizon is well-defined and bounded.

Note that the vector field of the closed-loop system (9) is continuous in τ and continuously differentiable in $\bar{x}$ in an open neighborhood of the initial data. Hence, the closed-loop system has a unique solution on a small interval $[0, \tau_s)$. Let $[0, \tau_f)$ be the maximal existence interval on which the unique solution exists. When $\tau_f < +\infty$, it means $\lim_{\tau\to\tau_f}\|\bar{x}(\tau)\| = +\infty$. When $\tau_f = +\infty$, all states of the closed-loop system are well-defined on $[0, +\infty)$.

Observing from (14) and the fact that $V_{\xi_i} \leq \frac{1}{2}D_{\lambda,n}^2(\xi_i) + \theta_i$ for $\theta_i > 0$, we see constants $C > 0$ and $\theta^* \geq 0$ exist such that

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} \leq -CV + \theta^*,$$

which immediately gives the boundedness of V on $[0, \tau_f)$.

From the definition of V, it is immediate to see ξ_i's are all bounded. Noting $\xi_1 = \bar{x}_1$ in (12) and $\alpha_1(\cdot)$ in (11), we learn the boundedness of α_1. Then following a recursive manner, we can obtain the boundedness of α_i's, actual control input u, and states $\bar{x}_i$'s on $[0, \tau_f)$ from $\xi_i = \bar{x}_i - \alpha_{i-1}$ in (12).

Since we have obtained the boundedness of u and all states of system (9), it directly follows that $\tau_f = +\infty$.

By recalling $\lim_{\tau\to+\infty}\gamma(\tau) = +\infty$, it is clear from the state scaling (8) and the bounded $\bar{x}_i$'s that $\lim_{t\to T} e_r(t) = \lim_{\tau\to+\infty} e_r(\mu^{-1}(\tau)) = 0$ and $\lim_{t\to T} x_i(t) = \lim_{\tau\to+\infty} x_i(\mu^{-1}(\tau)) = \zeta_i(T),\ \ i = 2,\ldots,n$. ■

By inspection, one can see the intelligible estimates for control design (see the proof of Proposition 1) and the concise analysis in prescribed-time performance exposure. This is mainly attributed to two aspects:

(i) The temporal transformation and state temporal scaling are performed before entering into the control design. By doing so, the *prescribed-time tracking* in finite t-horizon has been connected with the *boundedness* of the converted system (9) in infinite τ-horizon. This allows the advanced techniques tailored to boundedness analysis to be borrowed [1], [2], [3].

(ii) Delicate Lyapunov functions (13), instead of the conventional ones, e.g., $\frac{1}{2}\xi_i^2$, are exploited, wherein the $D_{\lambda,n}(\cdot)$ with a sufficiently smooth absolute value is incorporated as the key component (see (13)). This enables indefinite terms in $\frac{\mathrm{d}V_{\xi_i}}{\mathrm{d}\tau}$ to be estimated by taking their absolute values and a little bit of smoothing (see, e.g., (16)). Naturally, completing the square is avoided.

VI. Further Studies

In this section, we want to show that with the proposed strategy, certain *uncertainties* and *unknown nonlinearities* (of input-matched type) can be accommodated:

$$\begin{cases}\dot{x}_i = x_{i+1} + f_i(x_{[i]}), & i = 1,\ldots,n-1,\\ \dot{x}_n = bu + g(x) + d(t), \end{cases}\tag{19}$$

where b, termed the control coefficient, is a nonzero constant with an *unknown sign* and an *unknown size*. Nonlinearities $f_i(\cdot)$'s are as in system (1), $g(\cdot)$ is an unknown locally Lipschitz function and $d(t)$ is a bounded piecewise-continuous disturbance.

Assumption 2: There is a known smooth nonnegative function $\bar{g}(x)$ and an unknown constant $\theta \geq 0$ such that

$$|g(x) + d(t)| \ \leq\ \theta\bar{g}(x).$$

By performing transformations (3) and (8), we can obtain a weakly time-varying $\bar{x}$-system having the same dynamics as (9), except the last dynamics:

$$\frac{\mathrm{d}\bar{x}_n}{\mathrm{d}\tau} \ =\ bu + g(x) + d(t) + \bar{x}_n\frac{\gamma'(\tau)}{\gamma(\tau)} - \gamma(\tau)\frac{\mathrm{d}\zeta_n}{\mathrm{d}\tau}.$$

Design the adaptive controller as follows:

$$u \ =\ N(k)\eta, \qquad \frac{\mathrm{d}k}{\mathrm{d}\tau} \ =\ D_{\lambda,n}(\xi_n)\eta, \tag{20}$$

where $N(k)$ is a Nussbaum function [1], [3] (e.g., $k^2\cos k$) and function η is picked as follows:

$$\eta = \mathrm{sgn}_{\frac{\lambda}{2},n}(\xi_n)\Big(\bar{g}(x) + \sigma\big(\bar{x}_n^2+1\big)^{\frac{1}{2}} + \big((\gamma\frac{\mathrm{d}\zeta_n}{\mathrm{d}\tau})^2+1\big)^{\frac{1}{2}} + \big(\rho_n^2+1\big)^{\frac{1}{2}} + |D_{\lambda,n}(\xi_{n-1})| + |D_{\lambda,n}(\xi_n)|\Big),$$

with ξ_i's and ρ_n the same as in (12).

Theorem 2: Consider uncertain nonlinear system (19) under Assumptions 1 and 2. For any initial data $x_0 \in \mathbf{R}^n$, controller (20) guarantees the same prescribed-time exact tracking and the associated convergence as in Theorem 1.

Proof: As before, we let $V_1 = V_{\xi_1}$ and $V_i = V_{i-1} + V_{\xi_i},\ i = 2,\ldots,n$ with V_{ξ_i} as in (13). Since unknown b and unknown $g(x)$ are only added to the last dynamics, the first $n-1$ steps of controller design for system (1) can be directly used. Thus, we merely show the n-th step here.

By the first $(n-1)$ steps control design in Proposition 1, we have

$$\frac{\mathrm{d}V_{n-1}}{\mathrm{d}\tau} \leq -\sum_{j=1}^{n-1} D_{\lambda,n}^2(\xi_j) + |D_{\lambda,n}(\xi_{n-1})D_{\lambda,n}(\xi_n)|. \tag{21}$$

Then, noting V_{ξ_n} in (13) and ρ_n in (12), we can obtain

$$\frac{\mathrm{d}V_{\xi_n}}{\mathrm{d}\tau} = D_{\lambda,n}(\xi_n)\Big(bu + g(x) + d(t) + \bar{x}_n\frac{\gamma'}{\gamma} - \gamma\frac{\mathrm{d}\zeta_n}{\mathrm{d}\tau} - \rho_n\Big). \tag{22}$$

Now estimate the indefinite terms in (22) as follows

$$\begin{cases} D_{\lambda,n}(\xi_n)\big(g(x)+d(t)\big) \leq \theta\bar{g}(x)|D_{\lambda,n}(\xi_n)| \leq \theta\frac{\mathrm{d}k}{\mathrm{d}\tau},\\ D_{\lambda,n}(\xi_n)\bar{x}_n\frac{\gamma'(\tau)}{\gamma(\tau)} \leq \sigma\big(\bar{x}_n^2+1\big)^{\frac{1}{2}}|D_{\lambda,n}(\xi_n)| \leq \frac{\mathrm{d}k}{\mathrm{d}\tau},\\ -D_{\lambda,n}(\xi_n)\gamma(\tau)\frac{\mathrm{d}\zeta_n}{\mathrm{d}\tau} \leq \big((\gamma\frac{\mathrm{d}\zeta_n}{\mathrm{d}\tau})^2+1\big)^{\frac{1}{2}}|D_{\lambda,n}(\xi_n)| \leq \frac{\mathrm{d}k}{\mathrm{d}\tau},\\ -D_{\lambda,n}(\xi_n)\rho_n(\cdot) \leq \big(\rho_n^2+1\big)^{\frac{1}{2}}|D_{\lambda,n}(\xi_n)| \leq \frac{\mathrm{d}k}{\mathrm{d}\tau}. \end{cases}$$

Putting the estimates into (22) and combining (21), we see

$$\frac{\mathrm{d}V_n}{\mathrm{d}\tau} \leq (bN(k)+\theta+4)\frac{\mathrm{d}k}{\mathrm{d}\tau} - \sum_{j=1}^{n} D_{\lambda,n}^2(\xi_j).$$

Then, by use of Lemma 5 in [3], we can obtain the boundedness of $k(\tau)$ and V_n, which in turn gives the boundedness of ξ_i's. With the bounded ξ_i's, following the same argument in the proof of Theorem 1 directly yields the same prescribed-time exact tracking and associated convergence of x_i's. ■

VII. A Simulation Example

To validate the effectiveness of our prescribed-time control scheme, we consider the following robotic manipulator:

$$J\ddot{\rho} + B\dot{\rho} + G\sin\rho = u, \tag{23}$$

where ρ is the joint angular displacement of the manipulator; u is the torque to be applied; and J stands for the inertia, B for the parameter related to Coriolis and centrifugal forces, and G for the coefficient of the gravitational torque.

For system (23), by letting $x_1 = J\rho$ and $x_2 = J\dot{\rho}$, we can readily obtain its state space model, i.e., the second-order scenario of original system (1).

We first perform temporal transformation (3) to shift the original system to the one in infinite horizon and apply state scaling (8) for controller design. After that, we pick $D_{1,2}(s)$ with $Q_{\frac{1}{2},2}(s) = -\frac{1}{2}s^4 + \frac{3}{4}s^2 + \frac{1}{2}s - \frac{3}{32}$ and the associated $\mathrm{sgn}_{\frac{1}{2},2}(s)$ with $P_2(s) = 12s^5 - 10s^3 + \frac{15}{4}s$.

Following a recursive manner, we can derive the prescribed-time controller u as in (10). But for practical implementation, we convert it into the finite t-horizon by $\tau = \mu(t)$.

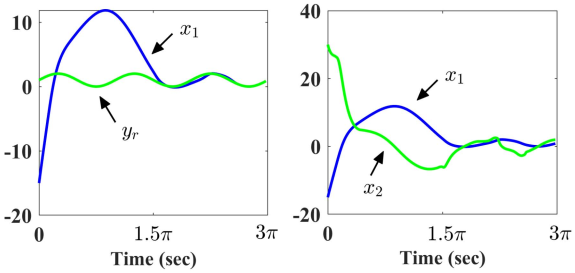

Fig. 1. Evolution of the system states.

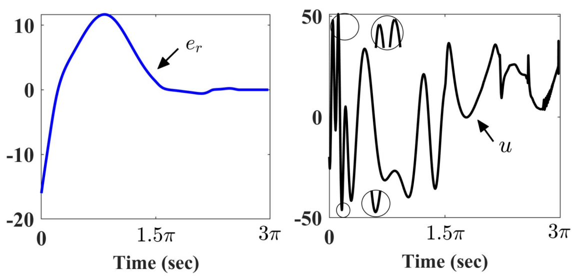

Fig. 2. Evolution of tracking error and control input.

Let the prescribed time $T = 3\pi$ and pick $\sigma = 0.8$ and $\mu(t) = 1.5\tan\frac{t}{6}$ and in turn $\gamma(\tau) = \mu'(t)|_{t=\mu^{-1}(\tau)} = \frac{1}{4}\sec^2(\arctan(\frac{2}{3}\tau))$. Let system parameters $J = 1$, $B = 1$, $G = 30$, select initial values $x_1(0) = -15$, $x_2(0) = 30$, and reference signal $y_r = \sin 2t + 1$.

Figs. 1 and 2 display that all signals (states x_1, x_2 and control input u) are bounded over the prescribed time interval $[0,\ 3\pi)$. Furthermore, state x_1 tracks reference signal y_r at about $T = 2.6\pi$ in Fig. 1 (or more intuitively, tracking error e_r in Fig. 2 converges to zero at about $T = 2.6\pi$). The simulation results illustrate the effectiveness of our control scheme.

VIII. Concluding Remarks

This letter has addressed prescribed-time exact tracking (PTeT) for a class of (uncertain) nonlinear systems. Despite the high nonlinearities that *distribute in each channel*, the system is still somewhat restrictive: except in the input channel, the nonlinearities have to be precisely known and no uncertainty (whether parameterized or not) can be allowed. This exposes the limitation of the control scheme in this letter. Therefore, it would be of significance and interest to develop new control schemes able to deal with a wide range of uncertain nonlinear systems.

Appendix

We would like to give two important lemmas drawn from [2], [3] to support our main results.

Lemma 1 [2], [3]: The following defined pseudosign function $\mathrm{sgn}_{\lambda,n}(\cdot)$ is $\mathcal{C}^n$ on $\mathbf{R}$ (indexed by parameter $\lambda > 0$):

$$\mathrm{sgn}_{\lambda,n}(x) = \begin{cases} \mathbf{sign}(x), & |x| \geq \lambda, \\ P_n\left(\frac{x}{\lambda}\right), & |x| < \lambda, \end{cases}$$

where $P_n(\cdot)$ is a $\mathcal{C}^n$ function defined in [2], [3]. Clearly, $\lim_{\lambda\to 0^+}\mathrm{sgn}_{\lambda,n}(x) = \mathbf{sign}(x)$.

Lemma 2 [3]: The following defined pseudo-dead-zone function $D_{\lambda,n}(\cdot)$ is $\mathcal{C}^n$ on $\mathbf{R}$ (indexed by parameters $\lambda > 0$):

$$D_{\lambda,n}(x) = \begin{cases} (|x| - \lambda)\mathbf{sign}(x), & |x| \geq \frac{3}{2}\lambda, \\ Q_{\frac{\lambda}{2},n}(|x| - \lambda)\mathbf{sign}(x), & \frac{1}{2}\lambda < |x| < \frac{3}{2}\lambda, \\ 0, & |x| \leq \frac{1}{2}\lambda, \end{cases}$$

where $Q_{\frac{\lambda}{2},n}(\cdot)$ is a $\mathcal{C}^n$ function defined in [3]. It can be viewed as an approximation of the ideal dead zone function.

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